COMMON CORE STATE STANDARDS FOR MATHEMATICS

Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning , strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately) and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy).

1 Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between approaches.

2 Reason abstractly and quantitatively.
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize* - to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents - and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and-if there is a flaw in an argument-explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimations and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying the units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Illustrating the Standards for Mathematical Practice:
Model with Mathematics

Handout 2

Lunch Money Models[1]

Kate

Grades 1 and 2, September

Three days into the current school year, my first- and second-grade students spontaneously began to talk about something that surprised me.

I was doing the lunch count. Conveniently, lunches in our school cost $2. On Friday, Ezra brought in a $20 bill to pay for future lunches, and the children began to offer ideas about it. Ezra said that he thought that for the $20 he would be able to buy 10 lunches. Galen agreed. I asked how they knew, and another child offered that each lunch cost $2, so 10 lunches would cost $20. Someone else added that for $10 you could get 5 lunches. Jared asked how many lunches you could get for $5. I knew we had only moments before we had to leave the room for art class, so I quickly tried to record some of what they were saying on chart paper so we would not lose it. After a few moments of silence, Leah, who had been deep in thought, said, "Two and a half. You could get two-and-a-half lunches for $5." Then she giggled at the silliness of half a lunch.

As often happens, there was no time to finish this interesting conversation when it arose. We had to go to art class and would need to return to this discussion later.

We did not return to the discussion for about two weeks. I was trying to figure out how and when to reintroduce it to the class so that I could recapture their enthusiasm. I also wanted to be sure that there was an entry point for all of my first and second graders. I knew that most of my first graders were counting by 1s and would not yet be able to think about those chunks of 2 in a meaningful way. There had to be a way to help make this activity accessible to them and at the same time engaging to the second graders in the class who could not only count by 2s but had also expressed in that initial conversation some understanding of the relationship between the number of dollars and the number of lunches. **start here

Initially, I made two decisions regarding the discussion. One decision was that we would explore this during a time other than our math period so that it would have a different feel for the children. The other decision was that I would reintroduce the scenario by having students show what they already understood about the problem. I mulled over in my

[1]From *Patterns, Functions, and Change Casebook* by Deborah Schifter, Virginia Bastable, and Susan Jo Russell. Pp. 23-28. Part of the *Developing Mathematical Ideas* professional development series. Copyright © 2008, Pearson Education. Used by permission. All Rights Reserved.

head how they could do that, and I kept coming back to the same issues. It was still quite early in the year, and I wanted to make this an activity in which everyone could be successful. That meant not asking my first and second graders to write what they understood, nor to have a lengthy conversation. So, where would I start? It finally came to me that the place to start was with models. Students would use a representation, such as cubes or pictures, to demonstrate what they understood.

It has become clear to me over the past few years that when children use models to show what they understand, it serves several purposes. First, and for me foremost, is that it enables more children to express their understanding, and it gives more children a way to follow what others are saying. Because there are a variety of models that correspond to various learning styles, children can find the way to express themselves that fits best with how they learn. They also have representations other than spoken words to help them make sense of another person's thinking.

Having different models can also help us to "unpack" the mathematics that is being explored. By comparing two models and thinking about their similarities and differences, we can sometimes look more deeply at the underlying mathematics.

So last Wednesday I reintroduced the context. "The other day, Ezra brought in some lunch money, and he said, 'I know if I have this much money I can get these many lunches.' All sorts of people started shouting out, "Oh, I know if I had this much I could get . . .,' and so I started thinking about that whole idea of how much each lunch costs and how many lunches you would get if you had a certain amount of money. So my first question is what one lunch costs." The class answered me, "$2."

I asked Moira if she could show me with cubes how much one lunch costs. She put two cubes together and held that up. I then asked the others how Moira was showing the $2. I wanted to make sure that the children understood how the cubes were being used.

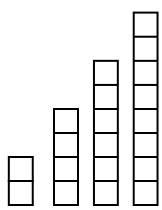
Abigail said that Moira was showing us the $2 that she could use to buy 1 lunch. When I probed further, she said that Moira put 2 cubes together. Leah's hand shot up and she added, "Each cube represents a dollar." What a great way to phrase it! I was so pleased to hear this coming from one of my second graders. This is one of the benefits of having students for two years. We talked about that for a minute because I wanted to reinforce the idea of the cube representing something else. I asked if I could use the cube to pay for the lunch (or for anything else). Of course they all said no. Sometimes my students think I am so silly when I ask questions like that, but I wanted to make the point.

Next I asked, if 1 lunch costs $2, then what would 2 lunches cost? I alerted the class to the fact that I had made a table to record the answers. Kayla said, "If 1 lunch costs $2 and 2 plus 2 is 4, then it must be that it's $4."

Number of Lunches	Number of Dollars
1 lunch	2 dollars
2 lunches	4 dollars
3 lunches	
4 lunches	
5 lunches	

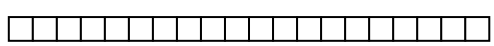
At this time, I introduced the rest of the activity. "I would like you to use the cubes to show me how much 1 lunch costs, 2 lunches cost, 3 lunches, 4 lunches, 5 lunches, all the way up to 10 lunches. I'd like you to, using Leah's word, represent how much the lunches would cost all the way up to 10 lunches using cubes." I made sure they all understood the task, and then I handed out cubes to pairs of children.

The children began their work quite enthusiastically. I was glad to see them getting right to work because I still was not sure that all the children were going to be engaged in the activity. As I began to see their models emerge, I was surprised at how many different models there were. I had expected to see many pairs making staircases that looked like this:



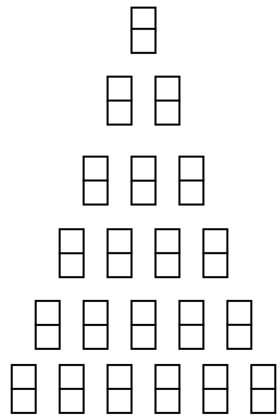
I had pictured each fact being shown separately: 1 lunch, 2 cubes; 2 lunches, 4 cubes. I actually only saw one pair of students, Abigail and Kayla, with a representation that looked like that.

I was surprised to see that Pedro and Galen had one long train of 20 cubes. When I asked how to see the cost of any number of lunches, Pedro pointed to the first 2 cubes and said that it was 1 lunch and $2. Then he pointed to those 2 and the next 2 and said that it was 2 lunches and $4. They had them all embedded into one train!



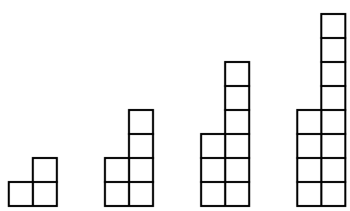
Moira and Justin made groups of 2 and then put the groups together. I heard Justin say, "You're always adding 2 more."

Eliana and Anna worked next to each other, but each ended up making her own model. Eliana made a pyramid of groups of two. It looked like this:



Interestingly, when we finally shared the models, Eliana had lost track of what in her model represented a lunch and what represented a dollar. I thought this was interesting because her construction looked so clear, but she had to think it through again to figure it out.

Anna's model had two trains for each relationship she was representing. For 1 lunch, she had a train of 1 cube and a train of 2 cubes. The first train represented the number of lunches, and the other, the number of dollars. I wonder if she was struggling with the same confusion as Eliana. When you have just one train (2 cubes), where is the lunch and where is the dollar? Anna avoided that problem by marking the dollars and the lunches separately.

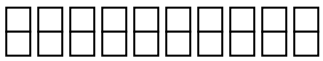


By the time we did our sharing on Friday, Micah and Thom, and Ezra all had representations that were like Anna's. In describing his and Thom's representation, Micah said that he always knew how many dollars for any number of lunches. He held each of a pair of trains in one hand and explained, "This (gesturing with the number-of-lunches train in his left hand) plus this (again gesturing with the same train) equals this" (referring to the longer train in his right hand that represented the number of dollars.) "Wow," I said, "Does that always work?" "I think so," he replied.

Charlotte worked with Ezra but had a different model. She showed me a train of 10 cubes and said that she knew it was $20. She said that she counted each one, 1, 2, 3 . . .10. Then she counted them again, starting at 11. When I asked her why she counted them again, she said, "I counted them again because each lunch was $2."

Jared slowly got around to putting cubes into pairs, and with some support, figured out what to do with them. Margot, his partner, was carefully making trains of 2.

Leah and Finn began by making trains of 2, each train to represent a lunch. They originally had them grouped so that there was a single train (1 lunch); then they added one train to show a group of 2 lunches; then added another train for a group of 3 lunches, and so on. By the end of their action, it looked like this:



As they kept working and making more trains of 2, it all became one group. When they shared their representation, Leah said, “Here is 1 lunch, here’s another lunch . . .” It had almost the same feel as Pedro and Galen’s. The 1 lunch did not need to be separate from the 2 lunches; the 1 was embedded in the 2.

Illustrating the Standards for Mathematical Practice: Model with Mathematics

Handout 3

From the CCSS-M document:

MP4: Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

What is modeling with mathematics at the K-5 level?

Modeling with mathematics at K-5. When given a contextual situation, mathematically proficient students express the situation using mathematical representations such as physical objects, diagrams, graphs, tables, number lines, or symbols. They operate within the mathematical context to solve the problem, then use their solution to answer the original contextual question. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Illustrating the Standards for Mathematical Practice: Model with Mathematics

Handout 4

Comparing Penny Jars[1]

This fourth grade class is describing and comparing two Penny Jars. Penny Jar A starts with 8 pennies in the jar; 2 pennies are added in each round. Penny Jar B starts with no pennies in the jar; 4 pennies are added each round. Students have worked with this comparison in small groups. Now the teacher puts a completed table showing the values for these two penny jars on the board, then asks what students notice about how the total number of pennies changes in the two jars.

Round	Penny Jar A: Total Number of Pennies	Penny Jar B: Total Number of Pennies
Starts with	8	0
1	10	4
2	12	8
3	14	12
4	16	16
5	18	20
6	20	24
7	22	28
10	28	40
15	38	60
20	48	80

Anna: Penny Jar A had 8 marbles more than Jar B. And then at the end, Jar B has a lot more than A.

Tonya: Penny Jar B catches up.

Derek: 4 is more than 2, so Jar B is going to have more.

Luke: It's bigger steps, so it will catch up.

Venetta: Jar A is slower than Jar B because of the 4. Jar B quickly catches up and passes.

Nadeem: By round 3, Jar B is just 2 away.

[1] Adapted from Russell, S.J., Economopoulos, K., Wittenberg, L., et al. *Investigations in Number, Data, and Space*®, Second Edition. Grade 4, Unit 9: *Penny Jars and Plant Growth*, pp. 158-159. Copyright © 2012, Pearson Education, Inc. Used by permission. All rights reserved.

Amelia: On round 4, they both have 16 pennies. And after that, Jar B has more.

Sabrina: Jar A was winning at first and then Jar B won because it got more each time.

Ramona: And also, the difference between 4 and 2 is 2.

Teacher: What do you mean, the difference between 4 and 2? Why is that important?

Ramona: The difference between what you put in is 2.

Teacher: And why is the 2 important?

Marisol: The difference keeps counting by 2s.

Andrew: Round 1, Jar A is in the lead with 6 more and Round 2, it's 4 more, and Round 3 it's 2 more.

Yuson: And then it's 0!

The teacher records what Andrew and Yuson are saying like this:

Round	Penny Jar A: Total Number of Pennies	Penny Jar B: Total Number of Pennies	
Starts with	8	0	
1	10	4	6
2	12	8	4
3	14	12	2
4	16	16	0
5	18	20	
6	20	24	
7	22	28	
10	28	40	
15	38	60	
20	48	80	

Emaan: It keeps on. If you look at the 2s thing, it keeps on going.

Teacher: So, if you look at it going forward, what is happening?

Richard: They are tied on round 4 and then it goes up by 2 the other way.

Teacher: What do you mean by it goes the other way?

Richard: Jar B goes up by 2 now. First it's 0, then 2, then 4, 6, and it keeps going.

Teacher: What about the graph of these two Penny Jars? What does that look like?

Noemi: The two graph lines get closer together and then they go apart.

LaTanya: It will look like an X. Once they've crossed over, they'll go wider and wider.

Teacher: And what does that mean?

Steve: First one has more. Then they have the same for just a minute. Then the other one has more.

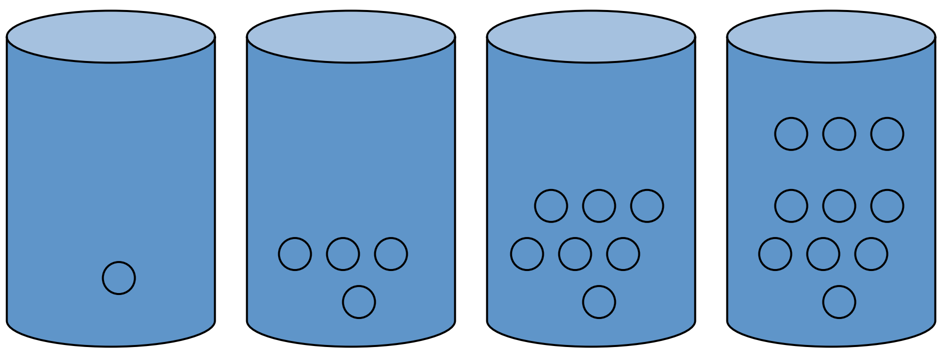
Bill: Jar B will have more and more. In the future Jar B will have a lot more pennies.

Illustrating the Standards for Mathematical Practice: Model with Mathematics

Handout 5

1, 4, 7, 10: What's the Same about The Penny Jar and the Staircase Towers?[1]

For this discussion in a first grade class, the teacher has put up a picture of four Penny Jars, containing 1, 4, 7, and 10 pennies, respectively, and a picture of a Staircase Tower with towers of 1, 4, 7, and 10. She first draws students' attention to the Penny Jars.



Teacher: I'd like you to take a look at the Penny Jar situation and let me know what you notice. What's happening?

Jacob: I notice that every day there are three more.

William: If you take one and add three more, that's 4, then add another 3 and that's 7, and if you add another 3 it would be 10.

Marta is shaking her head.

Teacher: Marta, how come you don't agree?

Marta: I don't know.

Teacher: What can we do to figure out whether William and Jacob are right?

Sacha: Count.

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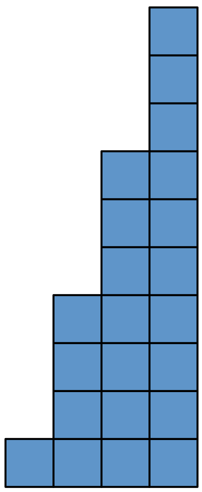
[1] Adapted from Russell, S. J., Economopoulos, K., Wittenberg, L., et al. *Investigations in Number, Data, and Space*®, Second Edition. Grade 1, Unit 7: *Color, Shape, and Number Patterns*, p. 148. Copyright © 2012, Pearson Education, Inc. Used by permission. All rights reserved.

The teacher works with the class to check whether there are three more pennies each day. To do this, they count on from 1 and the teacher holds up a finger for each count—2, 3, 4. Then they do the same thing to count on from 4 to get 7. The teacher also shows how many new pennies are in each jar by circling the pennies that were already in the jar the day before and then seeing how many more pennies there are. When students seem to agree that three pennies are added each day, she asks them to look at the Staircase Tower.



Leah: It's just like the penny jar cause it starts with 1 and steps up 3.

Tamika: I think it's skipping numbers. It's like 1, 4.

Allie: Hey, that's the same thing as the Penny Jar cause it's 1, 4, 7, 10.

Bruce: It's going to be 13, then 16, then 19. Oh! Wait a second! It's probably going to be the same thing for both of them.

Tamika: The Penny Jar is like the staircases except it's "add on," not "step up."

Nicky: Because they both start with 1 and add 3 or step up 3.

Teacher: Do you agree with Nicky that if you start with 1 and add 3, you will get the pattern, 1, 4, 7, 10?

Bruce: I think if you are in Kindergarten you might make some mistakes, but because we are first graders, we would get that pattern.

An important and pervasive idea in mathematics is that the same mathematics can model different situations. The number 3 can describe a quantity of marbles or a quantity of cars, even though marbles and cars are quite different from each other. Similarly, an expression like 3 + 5 can describe many different situations. Here, the number sequence 1, 4, 7, 10 can be generated from several situations because there is something mathematical that is the same about all those situations. Through their discussion, students are noticing how each situation starts at 1 and involves a constant rate of increase of 3.