COMMON CORE STATE STANDARDS FOR MATHEMATICS

The *Standards for Mathematical Practice* is a document in the CCSS that describes different types of expertise students should possess and mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education.

The first of these are the National Council of Teachers of Mathematics, NCTM, process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up:* adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately) and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy). This paper combines information from those sources and lists what students will be doing when they demonstrate mathematical proficiency.

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students:

- start by explaining to themselves the meaning of a problem and looking for entry points to its solution.
- analyze givens, constraints, relationships, and goals.
- make conjectures about the form and meaning of the solution.
- plan a solution pathway rather than simply jumping into a solution attempt.
- consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution.
- monitor and evaluate their progress and change course if necessary.
- who are older might, depending on the context of the problem:
  - ✓ transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need.
  - ✓ explain correspondences between equations, verbal descriptions, tables, and graphs.
  - ✓ draw diagrams of important features and relationships, graph data, and search for regularity or trends.
- who are younger might:
  - ✓ rely on using concrete objects or pictures to help conceptualize and solve a problem.
  - ✓ check their answers to problems using a different method.
- continually ask themselves, "Does this make sense?"
- understand the approaches of others to solving complex problems and identify correspondences between approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students:

- make sense of quantities and their relationships in problem situations.
- bring two complementary abilities to bear on problems involving quantitative relationships:
  - ✓ *decontextualize* - to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents and
  - ✓ *contextualize* - to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.

- use quantitative reasoning that entails habits of creating a coherent representation of the problem at hand:
  - ✓ considering the units involved,
  - ✓ attending to the meaning of quantities (not just how to compute them), and
  - ✓ knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students:

- understand and use stated assumptions, definitions, and previously established results in constructing arguments.
- make conjectures and build a logical progression of statements to explore the truth of their conjectures.
- analyze situations by breaking them into cases.
- recognize and use counterexamples.
- justify their conclusions, communicate them to others, and respond to the arguments of others.
- reason inductively about data, making plausible arguments that take into account the context from which the data arose.
- compare the effectiveness of plausible arguments.
- distinguish correct logic or reasoning from that which is flawed and, if there is a flaw in an argument, explain what it is.
  - ✓ Elementary students construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.
  - ✓ Later, students learn to determine domains to which an argument applies.
- listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve arguments.

4 Model with mathematics.

Mathematically proficient students:

- apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.
  - ✓ In early grades, this might be as simple as writing an addition equation to describe a situation.
  - ✓ In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community.
  - ✓ By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another.
- apply what they know to make assumptions and approximations to simplify a complicated situation, realizing that these may need revision later.
- identify important quantities in a practical situation.
- map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.
- analyze those relationships mathematically to draw conclusions.
- routinely interpret their mathematical results in the context of the situation.
- reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students:

- consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, protractor, calculator, spreadsheet, computer algebra system, a statistical package, or dynamic geometry software.
- are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations.
- who are in high school:
  - ✓ analyze graphs of functions and solutions generated using a graphing calculator.
  - ✓ detect possible errors by strategically using estimations and other mathematical knowledge.

- ✓ when making mathematical models, know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data.
- identify relevant external mathematical resources (e.g., digital website content) and use them to pose or solve problems.
- use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students:

- try to communicate precisely to others.
- try to use clear definitions in discussion with others and in their own reasoning.
- state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.
- carefully specify units of measure and label axes to clarify the correspondence with quantities in a problem.
- calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.
  - ✓ In the elementary grades, students give carefully formulated explanations to each other.
  - ✓ By high school, students have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students:

- look closely to discern a pattern or structure.
  - ✓ Young students might notice that three and seven more is the same amount as seven and three more or they may sort a collection of shapes according to how many sides the shapes have.
  - ✓ Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for the distributive property.
  - ✓ Older students, in the expression $x^2 + 9x + 14$, can see the 14 as 2 x 7 and the 9 as 2 + 7.
  - ✓ They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems.
- step back for an overview and shift perspective.
- see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. (e.g., They see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.)

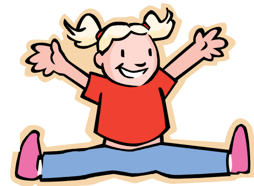
8 Look for and express regularity in repeated reasoning.

Mathematically proficient students:

- notice if calculations are repeated.
- look both for general methods and for shortcuts.
  - ✓ Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeated decimal.
  - ✓ By paying attention to the calculation of slope as they repeatedly check whether the points are on the line through (1,2) with a slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$.
  - ✓ Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead high school students to the general formula for the sum of a geometric series.
- maintain oversight of the process of solving a problem while attending to the details.
- continually evaluate the reasonableness of their intermediate results.

Problem of the Month

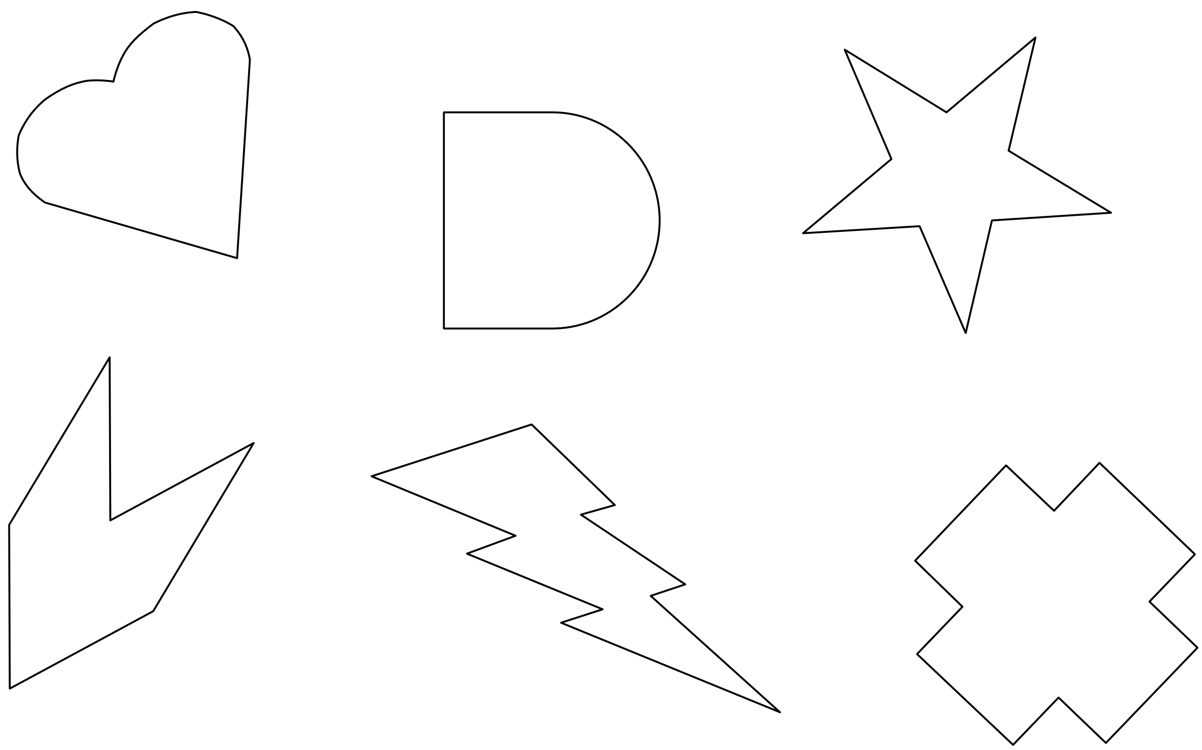
Part and Whole



Level A:

You and your friend have made a batch of cookies that have different shapes. You want to share each cookie between you and your friend so that you can taste each one. You decided you want to make sure to share the cookie so both pieces are the same. How should you cut your cookies to make sure each of you have the same shape and size of the cookies?

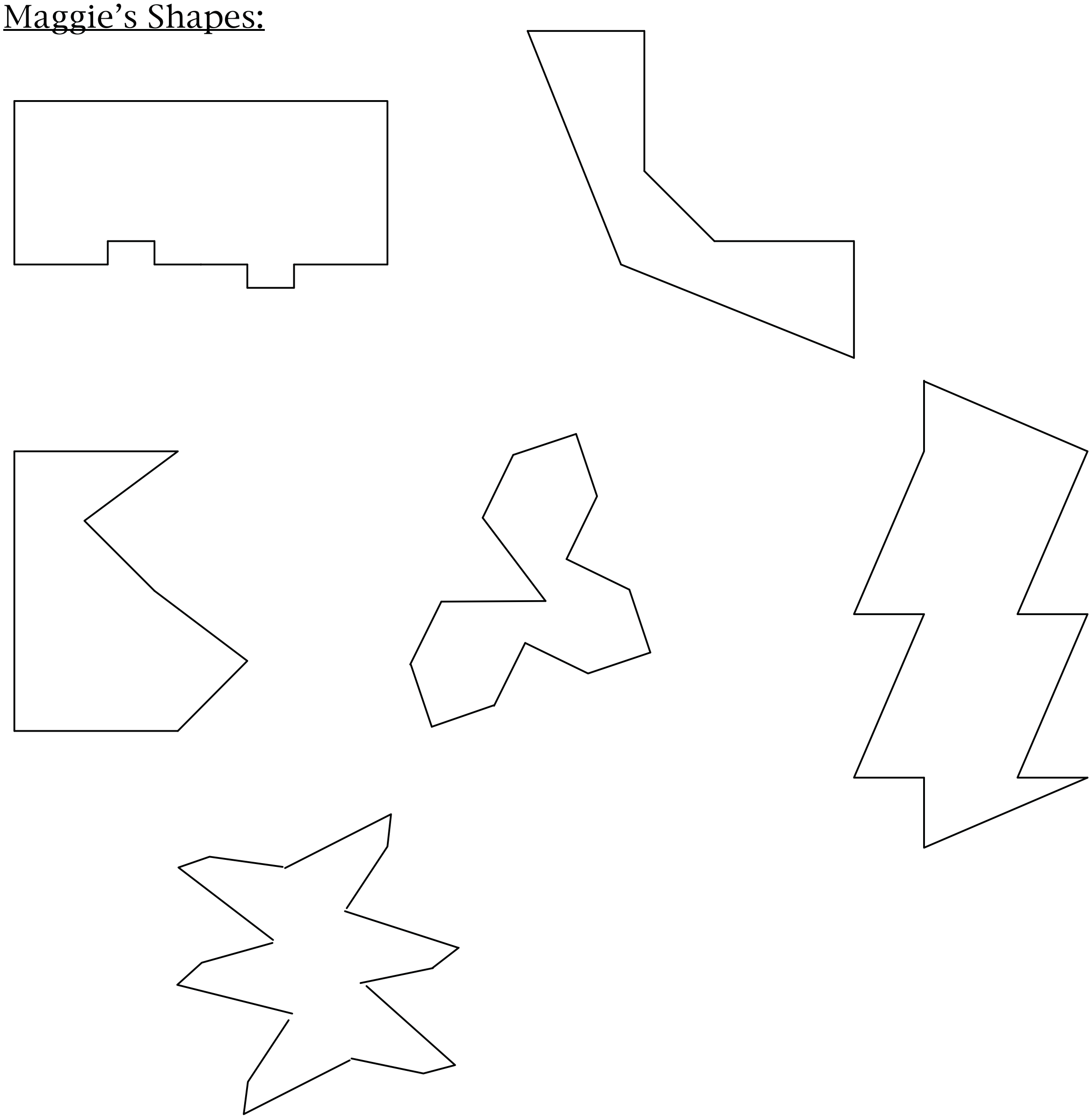
Draw a line through the cookies where you would make a cut and explain why the two pieces are the same.



Level B:

Maggie and Lexie were making funny shapes out of flat clay. They decided to play a game. Maggie would make a clay shape and Lexie would have to divide the clay shape using one cut-line. The two pieces would not have to look the same, but they would have to be the same size (same amount of clay). Below are Maggie's clay shapes. Show where Lexie should make a cut-line to make two pieces so both would be the same amount of clay.

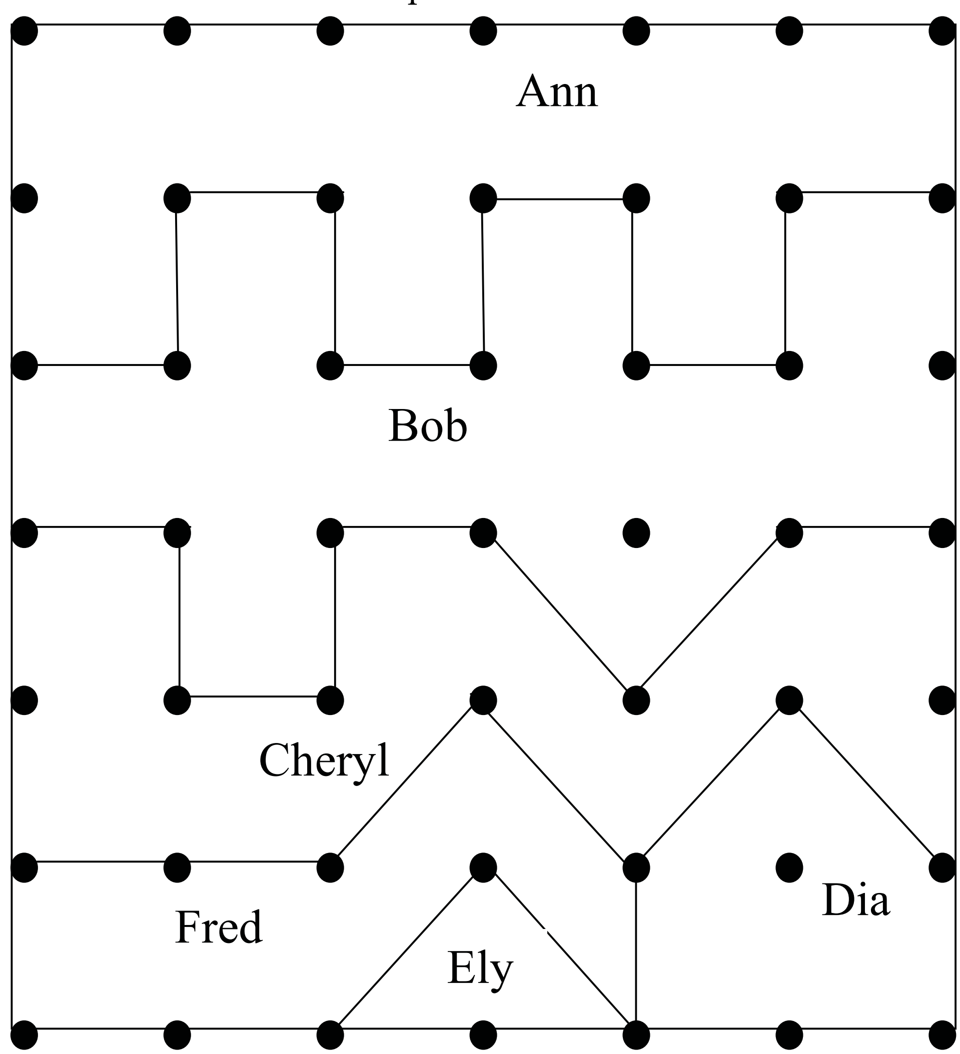
<u>Maggie's Shapes:</u>



Explain to Lexie why you know your methods are right.

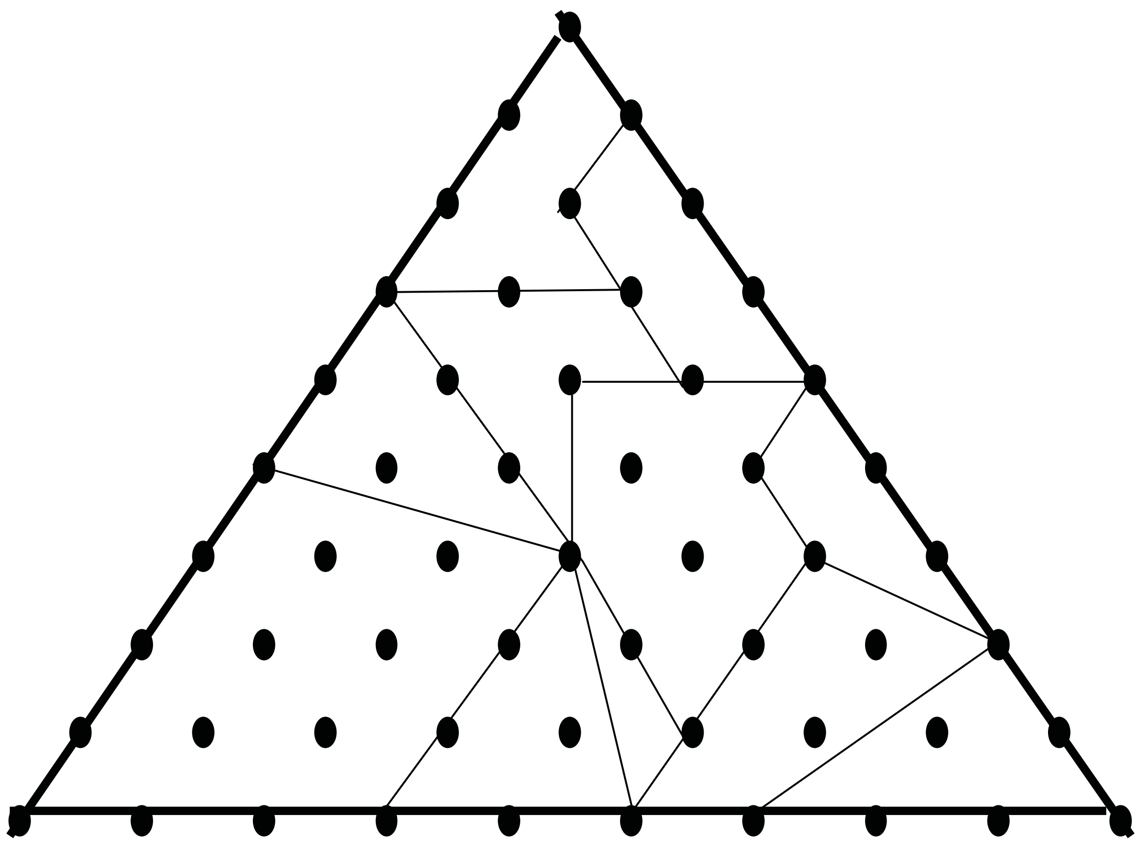
Level C:

Great Uncle Landowner has a parcel of land he owned. In his will, he left a map of the land that is divided into different regions. He wrote the names of each of his nephews and nieces on different regions of the map. He wrote the name in each region to indicate who will inherit that section of land. The regions range in size. Your job is to determine the fractional part of each region as it relates to the whole parcel. Examine the map below and determine the fraction piece of each region of land. Explain how you determined that fractional part awarded to each niece and nephew.



Level D:

You work for a puzzle company. You need to determine the fractional size of each piece so that company will know the materials needed for the different size pieces. They sent you the following puzzle. Determine the fractional size of each piece and explain your reasoning.



You have been assigned to create a more complicated puzzle. Create a design and provide a key to the fractional size of each shape, explaining how you determined its size.

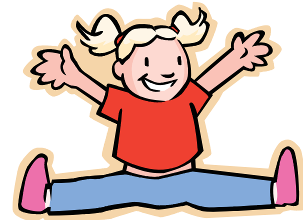
Level E:
A unit fraction has a numerator of one and a natural number denominator. Find five different unit fractions with a sum of 1.

Determine if there are more sets of five unit fractions, if so determine a general method for finding other sets. If not, prove why not.

What other *n* number of unit fractions can be found that sum to 1. Explain your reasoning and justify your conclusions.

Problem of the Month

Part and Whole



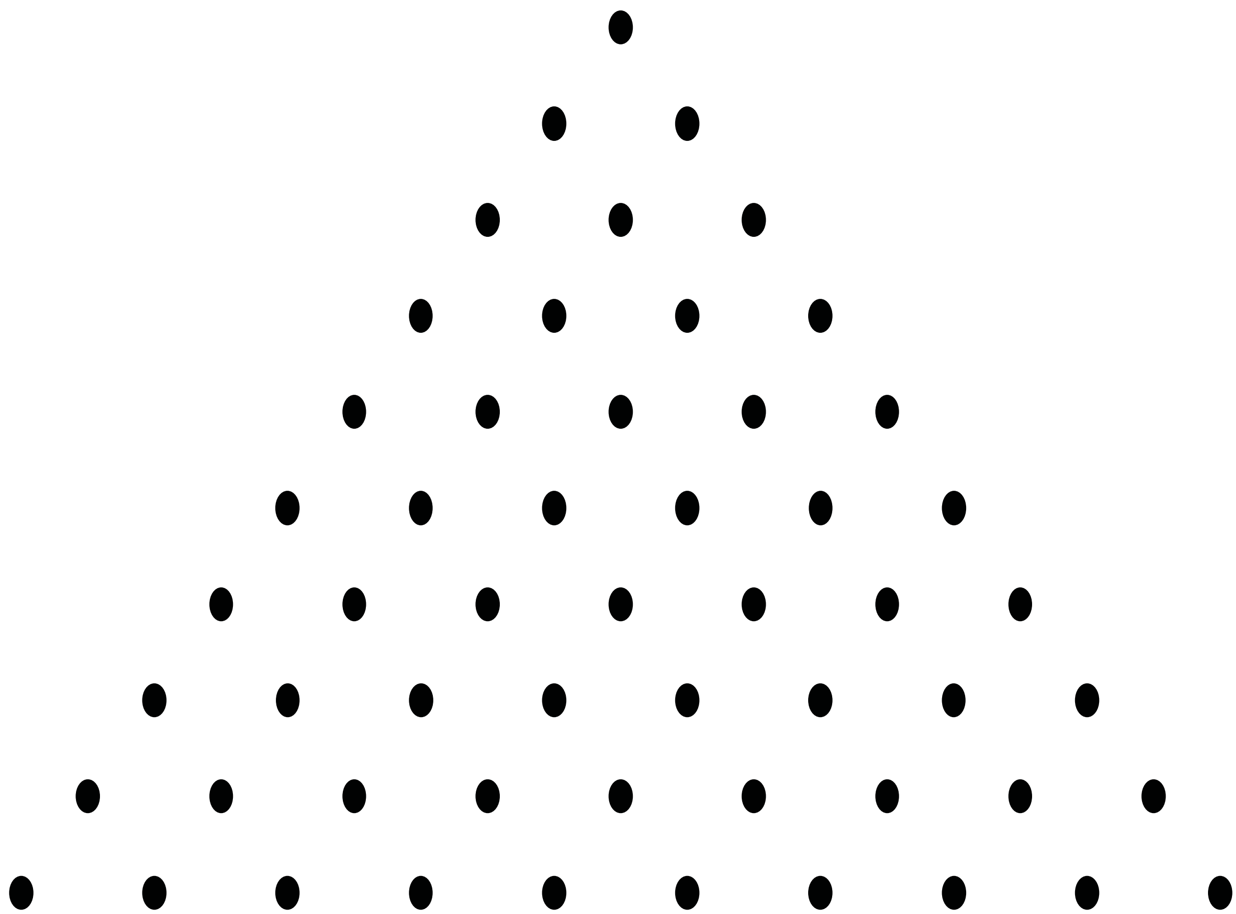
Primary Version Level A

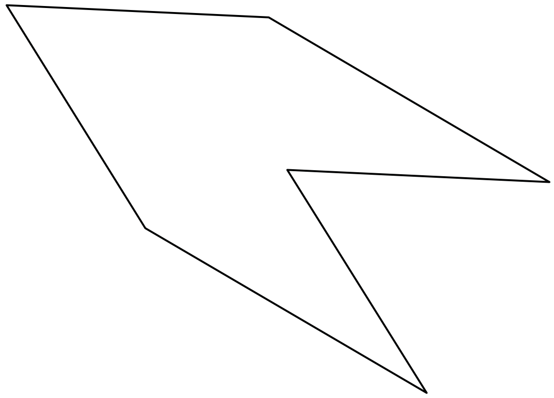
Materials: The enlarged paper with Cookies all different shapes, scissors, pencils and rulers.

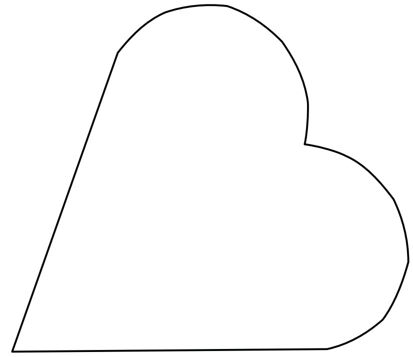
Discussion on the rug: (Teacher holds up the heart that was cut from shape.) "We want to share this cookie between two friends. How can we make one cut so each friend gets the same size piece?" (Students think about how to share the cookie. After soliciting some ideas from students (folding, drawing different lines, measuring, etc.), the teacher asks, "How will we know for sure?"

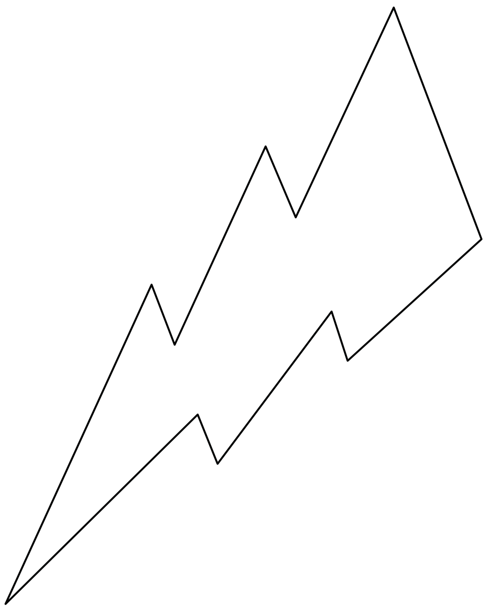
In small groups: (Students have enlarged cookie paper, rulers, pencils, tissue paper and scissors available)
Teacher says, "Here are different shaped cookies. You want to share each cookie, so you need to cut each one in half so the pieces are the same. Where should you cut it to make sure each one of you has the same size cookie? Draw a line to show where to cut." (Students draw a line to show where to make the cut. After the students are done, the teacher asks how we can show that both friends get the same amount. The class may actually cut out some of the shapes and cut them to show whether or not they are cut in half.)

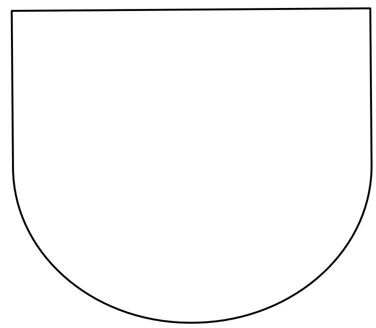
At the end of the investigation: (Students either discuss or dictate a response to this summary question.) "Explain how you know that both friends have the same amount after you cut the cookie?"

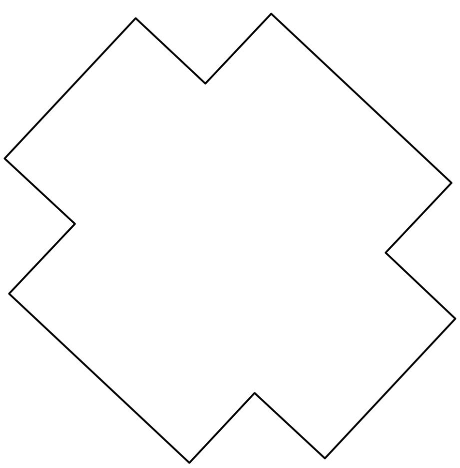


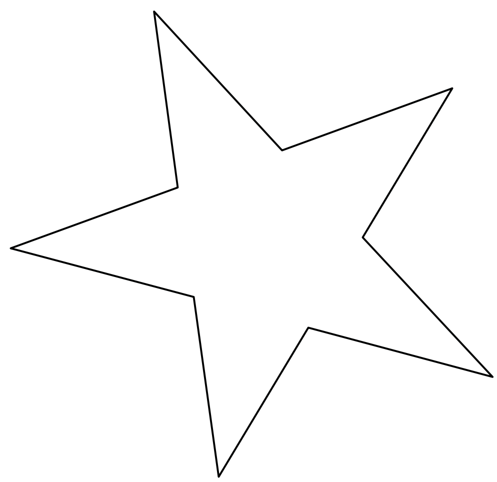












Guidelines for Problem of the Month

Why Problem of the Month

Problem solving is the cornerstone of doing mathematics. George Polya, a famous mathematician from Stanford, once said, "a problem is not a problem if you can solve it in 24 hours." His point was that a problem that you can solve in less than a day is usually a problem that is similar to one that you have solved before or at least recognized that a certain approach will lead to the solution. But in real life a problem is a situation that confronts you and you don't have an idea of where to even start. Mathematics is the toolbox that solves so many problems. Whether it is calculating an estimate measure, modeling a complex situation, determining the probability of a chance event, transforming a graphical image or proving a case using deductive reasoning, mathematics is used. If we want our students to be problem solvers and mathematically powerful, we must model perseverance and challenge students with non-routine problems.

How should the Problem of the Month be used?

The Problems of the Month are designed to be used schoolwide to promote a problem-solving theme at your school. The problem is divided into five levels, Level A through Level E, to allow access and scaffolding for the students into different aspects of the problem and to stretch students to go deeper into mathematical complexity. The goal is for all students to have the experience of attacking and solving non-routine problems and developing their mathematical reasoning skills. Although obtaining and justifying solutions to the problems is the objective, the process of learning to problem-solve is even more important. Administrators, teachers and parents should facilitate and support students in the process of attacking and reasoning about the problems. The self-analysis by students of how they went about approaching, exploring and solving the problems is a critical step in the development of becoming strong problem solvers.

The Problem of the Month is structured to provide reasonable tasks for all students in your school. Level A is designed to be accessible to all students and especially the key challenge for primary students grade K – 2. Level B may be the limit of where third and fourth grade students have success and understanding. Level C may stretch fourth or fifth grade students. Level D may challenge most fifth and sixth grade students and Level E should be challenging for most seventh and eighth grade students. These grade-level expectations are just estimates and should not be used as an absolute minimum expectation or maximum limitation for students. Problem-solving is a learned skill and students may need many experiences to develop their reasoning skills, approaches, strategies and the perseverance to be successful. The Problem of the Month builds on sequential levels of understanding. All students should experience Level A and then move through the tasks in order to go as deeply as they can into the problem. There will be those students who will not have access into even Level A. Educators should feel free to modify the task to allow access at some level.

One caution - the solution is not as important as the process of problem solving. Struggling to get started is a natural part of learning to problem-solve. The educator or parent should not be impatient with the student's struggle. In fact, encouraging and supporting the struggle with some frustration is exactly what the student needs. If a method is shown or told then the problem-solving process ends. Asking good but not guiding questions that require students to reflect and focus are most helpful. Having the student carefully read or be read the problem and then having the student restate the problem is often valuable. Having the student talk through the approach or the challenge is also effective in having the student rethink a strategy. Encouragement is the important key. In class, a teacher might have different students share their thinking (approach, not solution) with the class. The teacher should be careful not to assign value to the approaches. If students seem to follow an error in their process, the teacher should pose questions that make the class examine the process to uncover the error. A good problem-solver tries, fails, re-evaluates and tries again.

If some students are able to complete all five levels with detailed and accurate solutions and justifications, then the students may be challenged to go deeper into a problem or to extend it into another problem that is more complex. For example, in the Cutting a Cube problem, you may ask the student to formally prove the number of unique hexominoes. You could also extend the problem for them to explore septominoes (7 square figures) and develop number patterns for unique solutions of any nth-omino.

Role of the principal

The principal should embrace the concept of problem solving and model problem solving leadership. In that instructional leadership role, the principal should demonstrate being a facilitator of non-routine problems. The principal should begin by facilitating a session with the teachers in which they explore the Problem of the Month prior to presenting the problem to the students. The principal should model the same good attributes of instruction in problem solving with students. The same clarifying questions without guiding should be posed. In facilitating the non-routine tasks with the teachers, the principal should expect similar struggles and encourage teachers to reflect on the importance of disequilibrium and perseverance in the process. The teachers should follow this model as they present the problems to their students.

Once the problem is presented to the students in their classrooms, the principal should be visible in facilitating the tasks alongside the teachers. Visiting and/or leading a class as students share ideas and approaches with the other students encourages and empowers both teachers and students. The principal can also play a role in examining the student products with their teachers. Holding student write-up review meetings with teachers is an excellent way to focus on student understandings and their misconceptions. Another method would be for a principal to attend a class session where students are presenting their findings and solutions to the POM. Facilitating a presentation session is a great way to model how students can communicate and question each other's thinking. Encouraging deeper understanding and the justification of solutions are important mathematics outcomes of the POM process.

Summary Process

Often teachers use a summary or write-up format for the students to follow as they prepare to share their solutions and analysis of their problem solving process. Below is a sample format, which is appropriate for upper elementary grade students. Simpler versions are used for primary students.

Problem of the Month Write-up

- **Problem Statement**
  In your own words, state the general overall problem clearly enough that someone unfamiliar with the problem could pick up your paper and understand what you are asked to do.

- **Process**
  Describe in detail how you attempted to solve this problem. You may want to consider some of the following questions. You should also include things that didn't work.
  How did you get started?
  What approaches did you try?
  Where did you get stuck?
  Did you talk to anyone about the problem?
  Did talking to someone help or hinder you?
  What drawing, chart, graph, or model did you use?

- **Solution**
  State your solutions as clearly as you can. Include any charts, graphs, and lists and so on that you used to help you. If you were able to generalize the solution, include your results. Defend why you believe your solution is correct or the best possible answer. Your explanations should be written in a way that will be convincing to someone else.

- **Learning**
  Reflect on the problem. What did you learn? What mathematics did you use?

Including buy-in from the stakeholders

The students, parents and teachers are the chief stakeholders in the POM program. Obtaining commitment and buy-in by all is essential for a healthy and productive program. Assisting teachers with knowledge and skills is necessary to facilitate a strong learning experience. Informing parents of the program and supporting them with how they can help their students will make the program even more successful. Motivating students to want to problem solve and encouraging their perseverance will be the ultimate reward. The ultimate goal of math power for all students will never be realized if students aren't developed into strong problem solvers.

Below are sample letters to parents and teachers.

Dear Parents,

Problem solving is a fundamental goal of any strong mathematics program. Our school is committed to making each of our students a problem solver. It is never too early or too late to develop the real life learning skills of problem solving. Therefore, our school is embarking on a Problem of the Month program to help achieve this important goal.

Each month we will present our students with a non-routine problem for them to attack and solve. The problem will have several levels so that all students at our school will be able to work on a part of the problem appropriate to their learning development. All students should start with level A and work through the different tasks. It is understood that some students will not get too far into the problem. The process of attacking and struggling on a non-routine problem is important to learn. When your child has reached the maximum level of his/her understanding please celebrate their progress.

Trials, errors, and retries are key attributes of good problem solvers. We ask you to encourage your students to persevere. Many students might want to initially give up. The best support for your student is encouragement through good questions. Some good questions are*: What have you tried?, Why do you think it doesn't work?, Have you tried to make the problem simpler?, What do you need to know to be able to solve the problem?*. There are many other good questions; however leading or guiding questions are not helpful. The process of finding and understanding a solution outweighs the benefit of having a correct answer. Doing the problem for the student actually hurts the problem solving process. Many students will receive the hidden message that they can't solve problems by themselves and, will learn to stop and wait for someone else to answer.

You will play an important role in supporting your child's work on these problems. Once students have reached their level of understanding, they are asked to complete a write-up of their findings. Students should communicate how they went about solving the problem as well as the solution they found. This write up helps students understand how they think and approach new problems. We look forward to a partnership with you around problem solving. Thank you for supporting your child.

Sincerely,

Dear Teacher,

As you know, we are committed to improving our math program by emphasizing the importance of problem solving. We have made it one of our school wide foci. The Problem of the Month is a school wide program that we can all participate in to encourage and teach our students to be strong problem solvers.

Our first POM is titled "Cutting a Cube." It has five levels of complexity. Although all of our students should start at level A and work through the problem, some of our primary students will not be able to go much beyond the first level. That is okay; what is most important is the process and that students are stretched to go as far as their understanding and skills take them. At the same time we must encourage the students to struggle and persevere to develop their problem solving skills.

As a facilitator of POM, you must be careful not to lead or guide, but rather to pose clarifying questions and questions that require the students to reflect on their work. A good method is to have students from time to time share various processes they have tried. Be careful not to emphasize one solution method over another as students share their ideas. Don't have students share complete solutions until the conclusion and the summary presentations.

Many students might benefit from hands on experience in exploring the attributes of the cube or the process of actually cutting a cube apart. Paper cubes can be constructed with masking tape along the edges and students can use plastic knives to cut cubes into nets. When students are trying to determine which nets work and whether they have them all, a class might share a few examples of nets they have found and the teacher might pose a question like: *How might we classify the nets we found? How might we know when we have them all?*

You are encouraged to have your students follow the POM write-up. This provides a common format for examining student work with your colleagues. Processing the solutions and methods with your students is important in developing their skills.

Thank you for supporting your students in their development as math problem solvers.

Sincerely,

Teacher's Notes

Problem of the Month: **Part and Whole**

Overview:
In the Problem of the Month, *Part and Whole,* students explore rational numbers and solve problems involving symmetry, congruence, determining equal area, sub-dividing area models, reasoning about measurements, and generalizing about fractions. The mathematical topics that underlie this POM are understanding rational numbers through different representations. Students explore fractions through area models using symmetry, congruence, measurement and mathematical notation.

In the first level of the POM, students view different geometric figures and determine whether they can divide the figure into two identical pieces. Their task is to use symmetry to answer question of same shape and equal area parts. In level B, students are given a picture of flat geometric shapes made out of clay. Then the students are asked to find a way to make a straight line cut in order to divide the figure into two parts of equal amounts of clay. In level C, the students are presented with a rectangular map. The map is divided into six different regions of various sizes. The task for the students is to determine the fractional part of each region in terms of the whole rectangular area. In level D, students analyze a triangular region to once again find the fractional parts of the whole. The students are then asked to design their own map with sub-divided regions. In the final level of the POM, students are presented with an investigation to find five different unit fractions with a sum of 1. Students determine if there is more than one set of five unit fractions that sum to 1, if so they determine a general method for finding other sets. If not, they prove why not. They also explore other size sets of unit fractions that can be found to sum to 1.

Mathematical Concepts:
The mathematics involved in this Problem of the Month involves rational numbers. A rational number is a number that can be written as the quotient of two integers as long as the denominator is not zero. Rational numbers can be represented in numeric notation as fractions, decimals and/or percents. Students can visualize rational numbers in area models, as points on number lines, and as a comparison between two discrete sets.

Understanding Rational Numbers:
Rational number is an extremely important topic in mathematics. It is an area of mathematics that many students don't fully understand. Students who don't develop a conceptual understanding of fractions often get bogged down in the procedural rules that involve the operations of fractions. Fractions are used in many different ways. Fractions can be ratios of part to part or then again can be ratios of part to whole. Rates, ratios of different units, are often written as fractions, but operations on rates differ from operations on fractions. Many students don't think of fractions as a single quantity of which it is, rather students focus on the component numbers of the factions. Often students will focus merely on the numerator or the denominator when they perform

operations. Students are often confused with non-unit fractions, fractions with other than 1 in the numerator. Students often struggle with order fractions or placing them on a number line. Fractions greater than one requires increased understanding of fraction representation. Understanding the relationship of mixed numbers and improper fractions and when to use the different representations are often a challenge.

Proportional Reasoning
Proportional reasoning is a unifying idea that brings together several number operations and math concepts, and it's the basis for higher mathematics. Proportional reasoning involves multiplication, division, fractions, decimals, percents, ratios, rates, scale, proportions, probability, slope, similarity and linear relationships.

The Area Model for Understanding Rational Numbers
Area models are important tools for understanding rational numbers. Students should experience a variety of area models, with different whole shapes. Too often fractions are mostly illustrated as parts of circles and squares. An assortment of shapes, some non-standard and irregular will help students understand whole. Students can compare the size of regions using fractions. Sometimes one might compare one part of a shape to the whole shape, thus using fractions to compare the area of the piece to the area of the whole. Other times one can compare the size of one part to the size of a second part in a ratio of parts. In either case, a fraction is used to communicate the relationship.

Finding a fractional piece usually involves measurement. In dealing with some common fractions students can use symmetry and congruence to determine the size of a fractional part. For example, if a figure can be divided through symmetric lines into four congruent regions, then each sub-region is one fourth. Students begin to understand one-half through the ideas of symmetry and congruence. As students develop a deeper understanding of the part – whole relationship, they begin to use measurement skills. When symmetry and congruence are no longer sufficient for determining the size of sub-regions, student might try to decompose the figure into equal area pieces. Students may be successful by sub-dividing a figure to a set of equal size parts. By counting the number of equal size parts that fit in the area of the whole, students can determine a unit fraction of one of the parts. This is an effective strategy if a sum of the sub-region is equal to the area of the whole. Often though, the sub-regions do not add to the exact area of the whole. At that point students must find a way to compensate, by trying smaller sub-regions or to use a standard method of measuring the part and whole. This method involves comparing the size of the sub-regions using some unit of measure to the size of the whole using that same unit of measure. Some common misconceptions occur when students use different size units to measure the areas or don't make sure all the sub-pieces are an equal size. Other misconceptions involve misunderstandings about the part and whole relationship. For example different size objects may confuse students into believing that fractional parts are different even though the ratio of the sizes of the parts to the whole are the same. One fourth of a small circle looks a lot different than one fourth of a scalene triangle.

Fractions, Decimals and Percents

Ratios come in the forms of fractions, decimals and percents. Students use division of ratios to find decimal equivalences or write ratios as fractions to simplest forms. Students convert decimals and fractions to percents. Percent is a ratio having a denominator equal to 100. Students should understand that fractions, decimals and percents are different representations of the same ratio. Determining when to use each representation is an important aspect of proportional reasoning. Ratios are helpful in comparing large and small quantities. Reinforce students' understanding of fractions, decimals and percents by comparing parts to the whole.

Equivalent Fractions

Understanding equivalent fraction is extremely important. Many students don't fully understand that two equivalent fractions are the same quantity with just a different name. Students need to develop flexibility with equivalent fractions. Truly understanding equivalent names for fractions is the basis for adding two fractions with unlike denominators. Students who understand the equivalent fractions can use those properties to develop alternative methods to adding and subtracting fractions.

Ratio

A ratio is a relationship between two values that involves division. The form of a ratio may be a fraction a/b, a comparison of a to b or a:b, a rate (different units), a decimal or a percent. A ratio is the unit change (coefficient or slope) in a proportional relationship where $y = (b/a)\,x$. The scale size between two measures is a ratio. The slope of a line is the ratio between the distance on the vertical axis and the distance on the horizontal axis. Fractions are used in many different ways. Fractions can be ratios of part to part or then again can be ratios of part to whole.

Rate

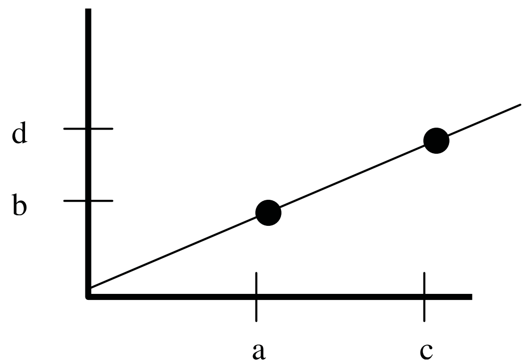
Rates, ratios of different units, are often written as fractions, but operations on rates differ from operations on fractions. Rate is a comparison of different units, such as miles per hour, unit price (items per cost), calories per serving, etc. When you add rates you follow a different algorithm than when you add fractions. Adding rates involves adding the numerators and denominators separately (add the same units together) then find the quotient. This is significantly different than the method for adding common fractions.

Proportion

Two ratios are proportional if they are equal. A proportion is a linear relationship involving two ratios. Often this relationship is expressed as direct variation, where $y = (b/a)\,x$. This relationship is also expressed as equivalent fractions $x/y = a/b$. In measuring remote similar objects, the ratio of the size of the two figures is in direct proportion to the ratio of the two distances the figures are from the viewing source. This proportional relationship is also true in perspective drawings. A scale factor and the distance from the original viewpoint govern the size of an object in a drawing.

A numerical proportion can be calculated given three values in a functional relationship. The function is a direct proportion where $y = (b/a)\,x$. Two values define the scale factor (b/a). The dependent value is found by multiplying the known independent value by the scale factor. A second method involves creating an equation of equal ratios, such as a to b is equal to c to d. This relation can be manipulated such that b to a is equal to d to c, or a to c is equal to b to d and its reciprocal. Students solve these relationships by graphing, cross-multiplying, determining a direct variation or guess and check.

A graph may be used to illustrate proportional relationships. Given any two proportional ratios, they lie on the same line between any one coordinate point and the origin. The points will lie on a line that intersects (0,0) and has a slope b/a. Any other point that is also proportional to those two points will lie on the same line.

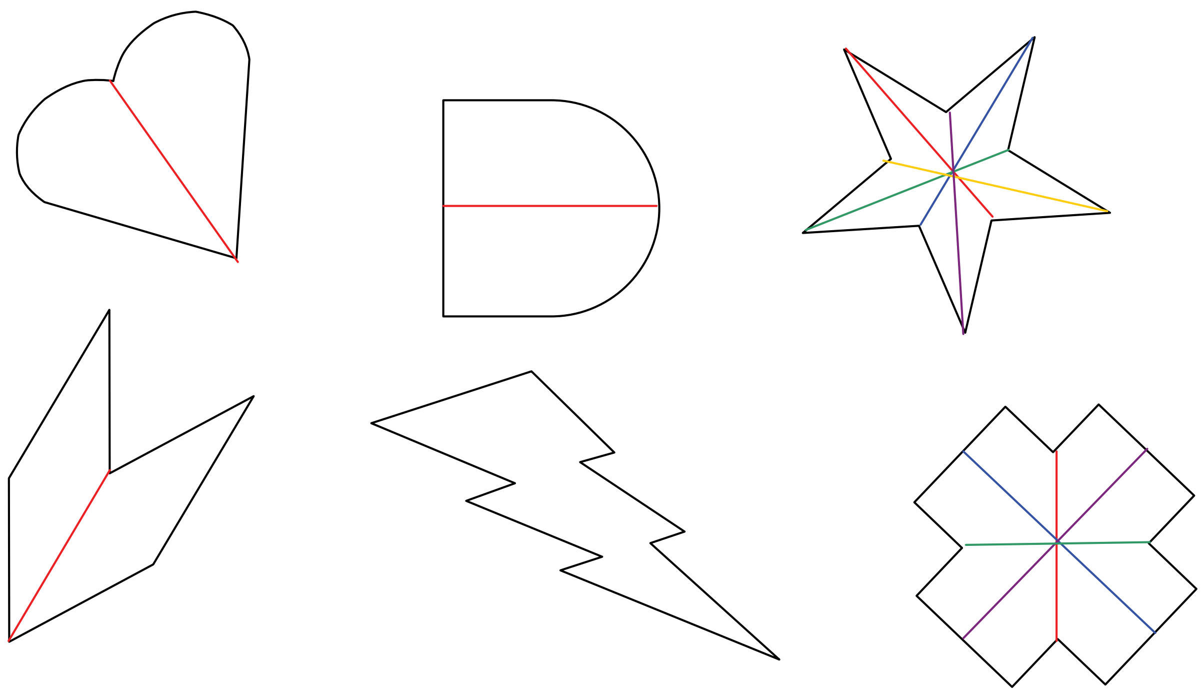


Scale
Scale is the ratio between two different measurements. The scale factor is that value by which the smaller measurement is multiplied to obtain the larger measurement. If a scale is written as the ratio of the smaller measurement to the larger measurement (a:b), then the scale factor is the reciprocal of that ratio, b/a. In a scale drawing, the size of each figure has a proportional relationship to the size of the original figure. The ratio of the size of the original figure to the size of the scale drawing is the scale factor.

Egyptian Fraction
An **Egyptian fraction** is the sum of distinct unit fractions, such as ½ + 1/3 + 1/16. That is, each fraction in the expression has a numerator equal to 1 and a denominator that is a positive integer, and all the denominators differ from each other. The sum of an expression of this type is a positive rational number *a*/*b*; for instance the Egyptian fraction above sums to 43/48. Every positive rational number can be represented by an Egyptian fraction. Sums of this type, and similar sums also including 2/3 and 3/4 as summands, were used as a serious notation for rational numbers by the ancient Egyptians, and continued to be used by other civilizations into medieval times. In modern mathematical notation, Egyptian fractions have been superseded by vulgar fractions and decimal notation. However, Egyptian fractions continue to be an object of study in modern number theory and recreational mathematics, as well as in modern historical studies of ancient mathematics.
Solutions:
Level A:

You and your friend have made a batch of cookies that have different shapes. You want to share each cookie between you and your friend so that you can taste each one. You decided you want to make sure to share the cookie so both pieces are the same. How should you cut your cookies to make sure each of you have the same shape and size of the cookies?

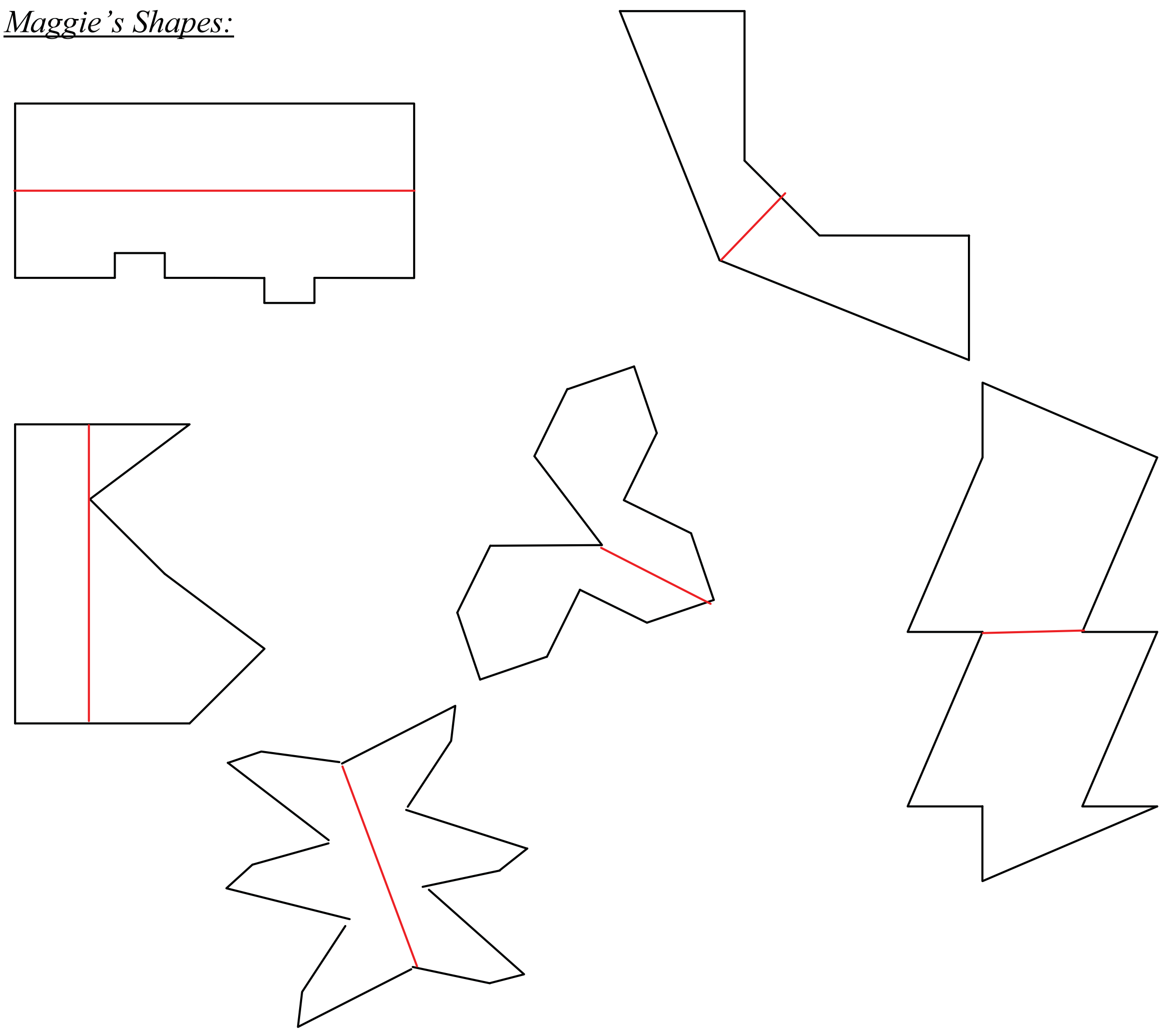
Draw a line through the cookies where you would make a cut and explain why the two pieces are the same.



The lines above divide the shape into two pieces that have the same shape and equal size (area). For the star and the cross there are multiple solutions. The star has five lines of symmetry and the cross has four lines of symmetry. The lightening bolt has no lines of symmetry so there is no straight line cut that can divide the figure into two pieces with the same shape and same size.

Level B:
Maggie and Lexie were making funny shapes out of flat clay. They decided to play a game. Maggie would make a clay shape and Lexie would have to divide the clay shape using one cut-line. The two pieces would not have to look the same, but they would have to be the same size (same amount of clay). Below are Maggie's clay shapes. Show where Lexie should make a cut-line to make two pieces so both would be the same amount of clay.

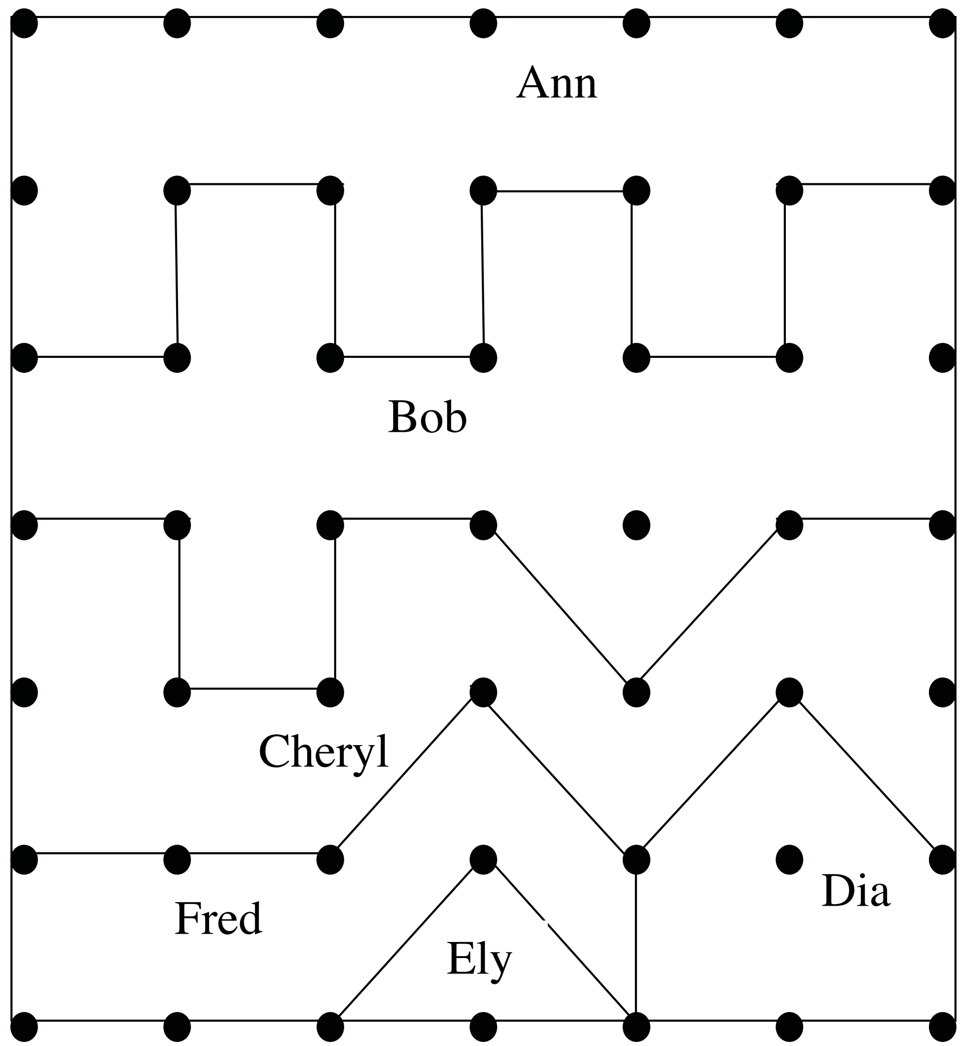
<u>Maggie's Shapes:</u>



Explain to Lexie why you know your methods are right.
The red lines show possible solutions for each of the six figures. Reading from left to right, the 2nd and 4th figures are symmetrical and the red line divides each into two congruent parts. The solution to the other four figures requires students to determine how the two pieces have equal area. The 1st figure has a notch which can be filled by the protruding piece making the two halves equal. Figure 3 requires the students to decompose the right half to see that the triangular shape can be cut and moved to the indention. Figure 5 is divided in half by the red line, because the two triangular shapes at the top and bottom are equal in area. The red line divides figure 6 into 2 congruent parts.

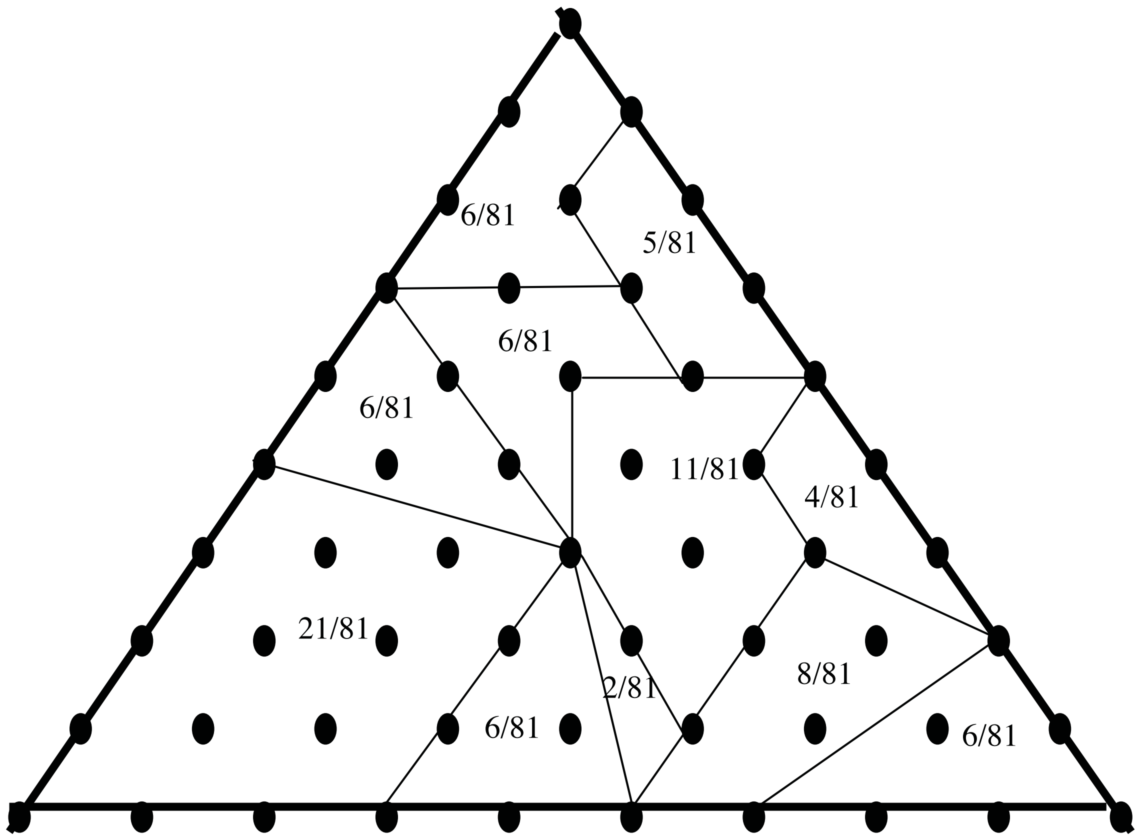
Level C:

Great Uncle Landowner has a parcel of land he owned. In his will, he left a map of the land that is divided into different regions. He wrote the names of each of his nephews and nieces on different regions of the map. He wrote the name in each region to indicate who will inherit that section of land. The regions range in size. Your job is to determine the fractional part of each region as it relates to the whole parcel. Examine the map below and determine the fraction piece of each region of land. Explain how you determined that fractional part awarded to each niece and nephew.



There are a total of 36 square units in the map of the land. So by counting the number of square units in each region, one can calculate the fractional parts of the inherited land. So, Ann gets 9/36 or 1/4, Bob gets 11/36 because the triangular piece is equal to one square, Cheryl's region is 8/36 or 2/9, Dia's region is 3/36 or 1/12, Ely gets 1/36 and Fred gets 4/36 or 1/9.

Level D:
You work for a puzzle company. You need to determine the fractional size of each piece so that company will know the materials needed for the different size pieces. They sent you the following puzzle. Determine the fractional size of each piece and explain your reasoning.



You have been assigned to create a more complicated puzzle. Create a design and provide a key to the fractional size of each shape, explaining how you determined its size.

If you add the rows (1+3+5+7+9+11+13+15+17=81) the large triangle is made up of 81 unit triangles. Counting up each unit triangle in a region is a strategy for determining the factional area of each sub-region. In cases where there are regions that do not contain partial triangular units, another strategy is needed. One other strategy is to determine a part that can be decomposed by finding a parallelogram region and then dividing that parallelogram into halves. In this manner, the parallelogram region can be measured and the triangle part will be half the area of the parallelogram.

Level E:
A unit fraction has a numerator of one and a natural number denominator. Find five different unit fractions with a sum of 1.

One common method is to find a number who has at least five divisors. One such number is 36. The factors are 1, 2, 3, 4, 6, 9, 12, 18, and 36. If you use any of those factors as the numerator over 36, the denominator, then you will have unit fractions. Next you try and find a set of the factors that sum to the denominator, 36. So one can find this set:

18/36+12/36+3/36+2/36+1/36 = 36/36
so 1/2+1/3+1/6+1/18+1/36 = 1

Determine if there are more sets of five unit fractions, if so determine a general method for finding other sets. If not, prove why not.

Using that single case, we can find additional sets, such as the one below:

18/36+9/36+4/36+3/36+2/36=36/36
so 1/2+1/4+1/9+1/12+1/18=1

Using that same process with another number with many factors, such as 48 you can find other sets of five. 48: 1, 2, 3, 4, 6, 8, 12, 16, 24

24/48+16/48+4/48+3/48+1/48 =48/48
1/2+1/3+1/12+1/16+1/48 = 1

What other n number of unit fractions can be found that sum to 1. Explain your reasoning and justify your conclusions.

Except for 1 and 1/2, fractions can be broken into unit fractions that sum to 1 using the following generalization,

$$1/a = 1/(a+1) + 1/(a(a + 1)) \quad \text{where } a \neq 0, -1$$

so using this formula through a recursive process n number of unit fractions can be generated starting by substituting a=1.
For example
1/1 = 1/2 + 1/2, now taking the second 1/2 fractions and substituting the denominator, 2, in for **a,** one gets 1 = 1/2+1/3+1/6. This can continue by substituting the next iteration (1/7 +1/42) for 1/6, generating a unit fraction sum of four addends, 1= 1/2+1/3+1/7+1/42.

Index of POM Task Sets

	Level A	Level B	Level C	Level D	Level E
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Cut it Out	Geometry - Spatial Reasoning	Geometry - Spatial Reasoning	Geometry - Spatial Reasoning	Geometry – symmetry and fractals	Geometry – symmetry and fractals
Cutting a Cube	Geometry - Spatial Reasoning	Geometry - Spatial Reasoning	Geometry - Spatial Reasoning	Geometry - Spatial Reasoning	Geometry - Spatial Reasoning
Digging Dinosaurs	Operations with Whole Numbers - Number sentences	Equations and Equality - Optimization	Equations and Equality - Optimization	Linear Equations and Functions	Non-Linear Equations and Functions
Diminishing Returns	Whole numbers – Basic Operations	Decimals – Adding	Ratios/Rates	Ratios/Rates and independent and dependent variables	Probability
Fair Games	Statistics and Probability - Fairness	Statistics and Probability - Fairness	Statistics and Probability - Fairness	Geometric Probability - Fairness	Geometric Probability – Expected Value and Fairness
Friend You Can Count On	Whole numbers – Basic Operations	Statistics and Probability - Combinations	Statistics and Probability - Combinations	Statistics and Probability	Statistics and Probability – expected value
Game Show	Game Theory	Game Theory	Game Theory	Game Theory – Theoretical Probability	Game Theory – Experimental and Theoretical Probability

Got Your Number	Whole Numbers – Basic Operations and Absolute Value	Whole Numbers – Basic Operations and Absolute Value	Fractions	Probability – Compound Probability	Probability – Compound Probability
Growing Staircases	Counting and Number Patterns	Series and/or Functions	Linear Functions – domain and range	Linear Functions – domain and range	Non-linear Functions
Measuring Mammals	Measurement and Transitive Property	Proportions and Scale Factor	Proportions	Proportions and Linear Functions	Proportions, Non-Linear Functions, Similar Triangles
Miles of Tiles	Number Sentences and Operations with Whole Numbers	Making a list and trial and error	Systems of Equations	Making a table and trial and error	Quadratic Equations-Factoring
Once Upon a Time	Time	Unit Conversions and variables	Whole Numbers – Basic Operations and Understanding the Remainder	Modular Arithmetic	Angles and Angular Measures
Part and Whole	Symmetry	Area	Proportions and Rational Expressions	Proportions and Rational Expressions	Fractions
Pick a Pocket	Gathering Data and Plotting Data	Statistical Measurement – Mean, Median and Mode	Statistical Measurement – Mean, Median and Mode and Normal Curve	Statistical Measurement – Mean, Median, Mode and Standard Deviation	Statistical Measurement – Mean, Median, Mode and Standard Deviation
Piece it Together	Geometric Shapes – Polygons	Geometric Shapes – Polygons	Platonic Solids	Platonic Solids	Platonic Solids
Rod Trains	Discrete Numbers – Assigning Values	Discrete Numbers and Combinations	Discrete Numbers and Combinations	Discrete Numbers and Combinations	Discrete Numbers and Combinations

Squirreling	Basic Operations with Whole Numbers - Subtraction	Understanding the remainder in division	Combinations and Making a table applying trial and error	Combinations and Making a table applying trial and error	Combinations and Non-Linear Equations
Shape of Things	Polygons	Polygons and Symmetry	Symmetry	Polygons	Polygons
Surrounded and Covered	Area using tiles	Area using tiles	Area	Area and Optimization	Fractals and Sierpinski's Tiangle
Through the Grape Vine	Gathering Data and Plotting Data	Statistical Measurement – Mean, Median and Mode	Statistical Measurement – Mean, Median and Mode	Line of Best Fit	Statistics – Application in a Case Study
What's Your Angle	Geometry - Polygons	Geometry - Polygons	Geometry - Polygons	Geometry – Polygons and Angles	Geometry – Polygons and Angles
Wheel Shop	Basic Operations of Whole Numbers	System of Equations	System of Equations	System of Equations	Venn Diagrams