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The ***NCSM Journal of Mathematics Education Leadership*** is published at least twice yearly, in the spring and fall.

The editors of the ***NCSM Journal of Mathematics Education Leadership*** are interested in manuscripts that address concerns of leadership in mathematics rather than those of content or delivery. Editors are interested in publishing articles from a broad spectrum of formal and informal leaders who practice at local, regional, national, and international levels. Categories for submittal include:

- Key Topics in Leadership
- Case Studies
- Research Report and Interpretation
- Commentary on Critical Issues in Mathematics Education
- Professional Development Strategies

Note: The last two categories are intended for short pieces of 2 to 3 pages in length

Submission/Review Procedures

Submittal of items for this publication must include three hard copies and a disk copy in text format. Do not put any author identification in the body of the item being submitted, but do include author information as you would like to see it in the ***Journal***. Items submitted for publication will be reviewed by two members of the NCSM Review Panel and one editor with comments and suggested revisions sent back to the author at least six weeks before publication. Final copy must be agreed to at least three weeks before publication.

On the cover:

This issue's cover, created by Bonnie Katz, is a study in contrast between the buzz created by the op-art grid in the background and the organic forms of the primitive animals superimposed over it in white.

National Council of Supervisors of Mathematics (NCSM)

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Purpose Statement

The purpose of the NCSM Journal of Mathematics Education Leadership is to advance the mission and vision of the National Council of Supervisors of Mathematics by:

- Strengthening mathematics education leadership through the dissemination of knowledge related to research, issues, trends, programs, policy, and practice in mathematics education

- Fostering inquiry into key challenges of mathematics education leadership

- Raising awareness about key challenges of mathematics education leadership, in order to influence research, programs, policy, and practice

- Engaging the attention and support of other education stakeholders, and business and government, in order to broaden as well as strengthen mathematics education leadership.

Comments from the Editor

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This is my last issue as Editor of JMEL. I am very grateful to Kay Gilliland and Linda Gojak, the two NCSM presidents who invited me to take on this role. My JMEL editorial experience belied the "old dog, new tricks" axiom, with most of my new tricks provided by our very talented design and layout person, Bonnie Katz, and our resourceful printer, Jeff Christo.

I have been privileged to serve NCSM in this editorship. As a parting message, I thought I'd offer a few suggestions to you, my leader colleagues, suggestions that are as much hopeful as advisory.

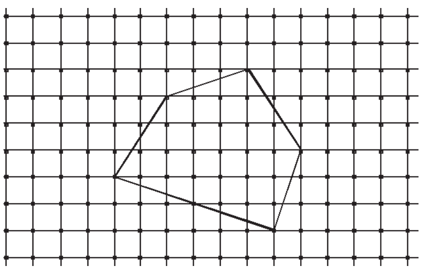
In your roles as mathematics leaders:

1. ***Think purposefully about teaching.*** Several decades ago, I taught in an alternative high school, situated in an old and austere warehouse in mid-city St. Louis. My colleagues and I designed the school to follow the vision and pedagogical principles in Carl Rogers' book, *Freedom to Learn*. Since a core principle extolled the value of interpersonal communication in both teaching and teacher education, my colleagues and I met each Thursday afternoon, after the students had left, to work through pressing teaching dilemmas. A local business had donated a video camera and monitor, so we watched weekly tapes of each other's classrooms. Brave or naïve, or both, we chose to watch the most challenging of our classes. Over several years, I managed to traverse several critical teaching thresholds through the following kind of sequence:
   i. I choose a taped difficult section of a recent class;
   ii. We commence watching at our Thursday meeting;
   iii. After a few minutes, someone says "Stop the tape;"
   iv. He or she asks, "Why did you do (or say, or ask, or ignore) *that*?"
   v. I reflect on and talk about my purpose, which probably was implicit, not conscious, and which more often than not did not fit with the action I took;
   vi. Together, we explore other possible purposes and/or alternative instructional moves more consistent with purpose.

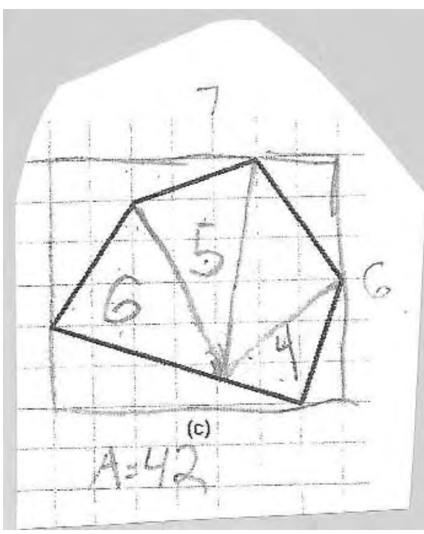
   The "why" in "Why did you do that?" related to purpose; the "that" related to the taped evidence. Thanks to those colleagues and our process, and that primitive old taping system, I developed over time an invaluable teaching habit of mind related to *purposefulness* and *evidence* which, once I left the classroom, has served me well in professional development and leadership work. Sadly, in that work, I have come to realize how few are the opportunities for most mathematics teachers to develop similar habits of mind. That, I believe, is where you as leader come in — particularly if you are a mathematics coach or a mentor of teachers or teacher leaders. "Why would you do that?" should be integral to teacher planning, and "What does the evidence tell you?" should be integral to lesson analysis. I hope you can be available to ask such questions, helping teachers to habituate them in their practice.

2. ***Think broadly about the evidence in student mathematical work.*** When the evidence being analyzed is student mathematical work — written, heard, or observed — it is very important to attend to potential along with deficit. Over the past decade, my colleagues and I have undertaken several initiatives in which analysis of student work on mathematics problems has been a central driver of professional development, most recently, an NSF-funded project, "Fostering Geometric Thinking (FGT) in the Middle Grades." In these initiatives, we

have observed a consistent phenomenon, which likely will not surprise you: most teachers have trained themselves to see quickly where students need help — a result, no doubt, of the daily demands of teaching and grading dozens of students. Less honed, however, is teachers' attention to the potential for productive mathematical thinking in students' problem solving work. It is an important complement. Take a quick example from our FGT field test. Students were asked to come up with multiple ways to calculate the area of this irregular pentagon:



One student's response seemed to have run out of steam in mid-solution:



At first glance, it is hard not to see only red lights flashing in the middle of the work, because in that part the student seems to be misusing a formula for area of triangles. Clearly, something needs fixing here. However, to focus only on the fixing would be short-sighted, since there are indications of at least two ways the student shows potential for reasoning with geometric relationships. First of all, the student has related the pentagon to a surrounding 6x7 rectangle, and that relationship can be exploited to help calculate the pentagon's area.

Second, the student has employed a potentially fruitful strategy in dividing the irregular pentagon into triangles. Unfortunately, the triangles chosen are not very helpful for precise calculations; however, with a few assessing and advancing questions from the student's teacher, the reasoning can proceed along a more productive path.

Developing the habit of noticing potential along with deficit in student efforts is an enormous challenge, but also enormously important. It is a reflective capacity that usually does not blossom on its own. Once again, the boost can come from leadership, in coaching sessions and in professional development engagements. Over time, in the lives of students, the experience of having their potential consistently nurtured can act to level the mathematical playing fields, which leads to:

3. ***Think selfishly about equity.*** A few years after leaving teaching in St. Louis, I directed a national "Study of Exemplary Mathematics Programs." One of the more memorable of the programs we studied was in a New England high school which, annually, walked away with honors in East Coast regional high school mathematics competitions. Significantly, year after year, each math team contained male and female students in approximately equal numbers, a noteworthy statistic in an era when gender inequities in mathematics were garnering national attention.

During our visit, we interviewed several young women on the current team, asking how they came to join the team. Each recalled being approached by the high school department head when they were in 7th and 8th grades, and how he talked up the high school math team as something they could aspire to. In other words, we realized, they were *recruited* well before high school — with all the boosting of self-image and motivation that individualized recruitment can provide!

Reading this, you might imagine that the high-school department head was politically progressive in his equity efforts. However, we saw neither politics nor progressivism in his strategies or actions. Self-interest, more than a sense of fairness, motivated him. The man wanted his teams to win, pure and simple. With that perspective, he knew he'd be a fool to ignore fully half the candidates available for the team.

It was a powerful lesson for me, which I have carried to this day. Sure, equity is about fairness. At the same time, and more compelling for me: equity is about self-interest and about capitalizing on our precious resources — the millions of students who may have been invisible, if not written off, in mathematics, but who will shine once they are tapped on the shoulder and recruited onto higher mathematical ground.

In closing, let me remind you that NCSM is a very special organization. Anyone who has participated in, or even watched, the bag-packing extravaganza that occurs before registration at each year's Annual Meeting, knows that the engine of this organization is fueled by the generosity, commitment, and energy of its members. Long may that engine run. And you can help it run by submitting articles to this journal and/or by offering to review manuscripts.

Teachers Need to Sell Mathematics Teaching:
Reaching Out to Excellent High School Students

Alice F. Artzt and Frances R. Curcio, with Naomi Weinman
Queens College, CUNY

Why is there a critical shortage of mathematics teachers? Is it because students who excel in mathematics have many lucrative opportunities in business, industry, research, and other related professions? Is it because the conditions under which many teachers function are often dismal, to say the least? Is it because many parents (even those who are teachers) recommend against teaching for their children? And is it because teachers often direct their excellent mathematics students into other professions? Of course, the answer to all of the above questions is a resounding "yes!" We believe that since teachers have a remarkable influence over their students, they CAN make a difference in enticing their best students to consider a career in mathematics teaching, and we have designed a way to help by hosting an annual conference for high school students that celebrates mathematics teaching.

Our experience indicates that even students who love mathematics and excel in mathematics have often never seen mathematics teaching at its best. For this reason they never envision a career in mathematics teaching as interesting or exciting. Therefore, one of the primary purposes of a conference that "celebrates mathematics teaching" is to expose talented mathematics students to exciting lessons taught by inspirational and exemplary mathematics teachers. Such a conference provides a forum in which the speakers who are secondary mathematics teachers can "sell" their own profession — teaching! Each year the TIME 2000 Program (i.e., Teaching Improvements through Mathematics Education), a four-year, multi-faceted undergraduate program designed to recruit, prepare, and retain future secondary mathematics teachers[1], hosts just this type of a conference at Queens College of the City University of New York, called, "Celebrating Mathematics Teaching."[2] At its recent fourth annual conference, a record-breaking number of more than 300 high school students and 30 of their mathematics teachers as well as over 80 mathematics education undergraduates and 15 college faculty members were in attendance.

An inspirational keynote address was delivered by Cathy Seeley, President of the National Council of Teachers of Mathematics. Local and nationally-acclaimed dynamic mathematics teachers, many of whom are TIME 2000


NCTM President Cathy Seeley delivering her inspiring keynote address

[1] For more information about TIME 2000, please see Artzt & Curcio (2007), and visit www.qc.cuny.edu/time2000 .

[2] The Simons Foundation, the primary supporter for the Math for America Program, is gratefully acknowledged for funding the conference.

graduates and Queens College graduates, made exciting presentations that actively involved the conference participants, that is, the high school students, in mathematical investigations and explorations. More than 80 TIME 2000 students volunteered to prepare for and assist at the conference. Several TIME 2000 graduates returned not only to be on the program but also to participate in a panel discussion conducted by a TIME 2000 senior. The purpose of the conference, planning and implementing the conference, and effects of the conference on recruitment are discussed below.

Why Host a Conference?

Several years ago it became evident that the mathematics teacher shortage was a means of highlighting the great number of unqualified middle and high school teachers who were teaching mathematics. Unfortunately, horror stories far outweighed the Herculean efforts of the very qualified, capable, and dedicated mathematics teachers. That is what contributed to the notion of creating a conference in which the work of highly qualified, enthusiastic, and devoted mathematics teachers would be highlighted, and in fact, "celebrated." If it were indeed true that a vast number of middle and high school students were being taught by uncertified teachers, then perhaps they needed to experience some exciting lessons taught by inspiring mathematics teachers. Perhaps, such an experience would even make them consider becoming a teacher themselves! Giving mathematically capable students the opportunity to experience exciting lessons taught by local and nationally recognized enthusiastic, exemplary mathematics teachers who explicitly describe their love of mathematics teaching in a professional conference setting has proven inspirational and has shown potential as a powerful means of recruitment.

How Did We Make It Happen?

As we thought about our vision for a conference, we had several concerns. Would teachers want to give presentations even though they would not receive compensation? Would the school administrators be willing to release these mathematics teachers to participate? Would administrators and mathematics teachers find such a conference worthwhile for their students and allow them to participate? We wanted to have one mathematics teacher from each school select students who were good in mathematics and might potentially consider becoming teachers. But, would these teachers be permitted to escort the students to the conference? Would the students be able to get to the Queens College campus? Would we be able to get enough help in conducting the conference? Well, as they say in Hollywood, "If you build it, they will come!" As it turns out, the answer to all of the above questions was a resounding, "Yes!"

Selecting and inviting the teacher-speakers. We contacted the exemplary teachers who had worked for many years as cooperating teachers for our student teachers. We contacted past graduates who we knew were spectacular, passionate teachers. They were all flattered by our request to have them teach their favorite lessons. They were flattered at the idea of being "showcased." Their supervisors, the mathematics assistant principals and principals, were thrilled to have their schools highlighted in the program. And so, the teachers enthusiastically agreed to come. In fact, they thanked us for inviting them. One teacher sent the following e-mail at the conclusion of the conference:

> *The conference was wonderful! Thanks again for giving us the opportunity to present a session.*

Jacqueline Seenarraine, TIME 2000 graduate teaching at H. Frank Cary High School, with Tanica Meade, a TIME 2000 undergraduate

Some teachers have presented for several consecutive years. They absolutely love the chance to share their love of mathematics and teaching. Without exception, at the end of the day, each presenter offered to teach a lesson again the next year! In fact, because we want to reach out to other outstanding speakers, we regret when we do not invite them back.

Inviting the students and their teachers. The students and their teachers were equally excited to come. As it turns out, by our fourth year, teachers in the school were "fighting" for the chance to attend the conference with their students. As one mathematics department chairman reported,

> *Our students had a wonderful time at Queens College today. Eleana [a mathematics teacher] was like a kid again, all excited and rejuvenated! She could not say enough about the atmosphere! Thanks for providing our students and teacher with such a great experience.*

Obtaining help to prepare for and conduct the conference. All we had to do to find help in conducting the conference was to look within our own "family," our TIME 2000 students! They jumped at the chance to help. Some students stuffed folders. Other students made room signs. Some worked at the registration desk. Others escorted the high school students to their workshop rooms on campus. Others made sure the speakers had the proper materials. And, what was their reward? They attended the sessions themselves! They experienced the joy of feeling important by helping others who were younger and felt lost on a big college campus! And yes, we gave them an ice cream party at the end when all our guests left!

Student and graduate student panel discussion. One of the main features of the conference is a panel discussion conducted by the undergraduates and graduates of the TIME 2000 Program. As the high school students pick up their boxed lunches in return for submitting an evaluation of the conference, they reconvene in the auditorium to participate in a lively question-and-answer session regarding the TIME 2000 Program. The enthusiasm and passion exuded by members of the panel for choosing to teach mathematics as their profession permeate the auditorium.

Student Panel consisting of TIME 2000 undergraduates Shari, Randall, Samantha, Julio, Sarah and Ricky (all in blue shirts) and John Chae, TIME 2000 graduate teaching at Baldwin High School

Effects of the Conference on Recruitment

After the conference, several of the high school students came up to us and thanked us for giving them this wonderful day. One young lady, an honors student from one of the specialized high schools in science and mathematics in New York City, was considering applying to several Ivy League colleges, but said that she had such a wonderful time that Queens College, specifically the TIME 2000 Program, was now the only place to which she was going to apply. Other students who had never even considered mathematics teaching were so inspired by the conference that they were now rethinking their career options. In the words of one of the students:

> *I never really considered a career in math, but looking around and listening to these wonderful people really made me reconsider.*

After only four years, interest and enthusiasm for the conference have spread. With approximately 30% of the incoming freshmen indicating that attendance at one of the conferences influenced their choice in applying for the TIME 2000 Program, the effects of this recruiting strategy are starting to be manifested.

National attention continues to be focused on the poor performance of American high school students in mathematics (Rising above the Gathering Storm, 2006, p. A20; State of the Union, 2006, p. A19). With almost 60 percent of American eighth graders being taught mathematics "by teachers who neither majored in math nor studied it to pass a certification exam" (Schemo, 2006, p. A20), this is no surprise. Without mathematically competent teachers who understand how students learn and who employ instructional strategies that motivate students and engage them in meaningful learning, middle school and high school students will not be able to reach the proposed goals of the American Competitiveness Initiative (State of the Union, 2006). Although there may be many ways to solve the critical-shortage-of-highly-qualified-mathematics-teachers problem, innovative, home-grown solutions are needed to build an infrastructure to support our profession. If each of us in our own small way can use some of the experiences described here, to highlight the excitement, joy, and value of mathematics teaching to secondary school students, perhaps we can all contribute to increasing the pool of potential, exemplary mathematics teachers. Celebrating mathematics teaching and "selling" this profession to outstanding high school students has the potential to really make a difference!

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Making the "Cut":
One District's Strategy for Algebra Placement

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ABSTRACT:
Some of the most discussed issues in mathematics education today involve Algebra and its instruction. These issues include the optimal timeline for when students first take a formal algebra course, the related selection process for getting into that first course and what algebra instruction should generally look like throughout the curriculum. Algebra is being recognized as a key "gate-keeper" course for high school and college success and has even been called an emerging "civil rights issue" by some researchers and authors. When to place students into an algebra class and how to ensure that a student is ready for Algebra are both critical curriculum decisions for a district. In many districts, algebra placement is a process that may be undergoing considerable revision along with how algebra is integrated across the curriculum. This article describes one district's approach for evaluating and revising their placement strategy for admitting students into their first middle school algebra course.

"Not every child has an equal talent or an equal ability or equal motivation, but all children have the equal right to develop their talent, their ability and their motivation."

~ John Fitzgerald Kennedy, 1963

John Kennedy's famous civil rights quote that "all children have the equal right to develop their talent, their ability, and their motivation" was made in a speech to the American people in a radio address on the morning of June 11, 1963. That was the morning that President Kennedy sent in the Alabama National Guard to open up the University of Alabama to two well-qualified black students. Access to a college education, for all qualified students was of course one of the most important civil rights issues of that day. In many ways, that civil rights issue is still with us in mathematics education and is often represented within the discussions of when students take Algebra and how they study it throughout their K12 coursework.

In mathematics education, the timeline for when students take Algebra, the related selection process, and what algebra instruction should look like throughout the K12 curriculum are some of the most discussed issues in the profession today. For example, algebra instruction and placement have been strongly represented in the last several National Council of Teachers of Mathematics and National Council of Supervisors of Mathematics annual conferences, with numerous sessions and presentations dedicated to algebra instruction. Another example of this professional dialogue is the new 2006 document by the National Council of Teachers of Mathematics, called "Curriculum Focal Points" which details topics of particularly important focus for pre-kindergarten to grade 8 mathematics instruction. This document has algebra well identified as a focus area, with consistent references to "number operations and algebra" as focal points from first grade through fifth grade, and an emphasis on "algebra" itself as one of the key focal points in grades 6-8. Algebra is obviously continuing to become an ever more important topic in K12 mathematics instruction.

The importance of algebra is also increasing as computer technology impacts the ways in which we have to teach mathematics (Heid, 2005; Hegedus & Kaput, 2004). Instructional tools such as graphing calculators, computerized algebra programs and homework helping websites are allowing schools and teachers to more effectively provide the instructional depth to algebra that it deserves in its growing importance in the K12 mathematics curriculum (Heid & Edwards, 2001). In fact, professional associations such as the Association of Mathematics Teacher Educators are commonly mentioning algebra as an instructional area particularly compatible with new technologies of instruction (Association of Mathematics Teacher Educators, 2006).

In a direct reference to the civil right passions of the 1960's, algebra has even been called an emerging "civil rights issue" for the next decade (Checkley, 2006; Moses, 2000; Moses, 1994). From a research perspective, an early understanding of algebra has been shown to be a key (and perhaps THE key) predictor for success in high school mathematics coursework and even entry into college (Burris, Heubert, Levin, 2004). A study by Horn and Nunez (2000) illustrates the importance for students in taking the advanced mathematics coursework that follows an early algebra placement. In their study, students of parents who never attended college more than doubled their chances for enrolling in a four-year college when taking coursework past Algebra 2. A well-prepared student that gets into an "early algebra sequence" may well have a distinct academic advantage over a student who does not get into that sequence. In addition, a poorly prepared student who fails at an early algebra course, may well be doomed to struggling in mathematics or even discarding mathematics as something that they are only minimally interested in learning (Schoenfeld, 2002).

Thus, how a school district selects students to enter a formal algebra course and when that selection process occurs is becoming critically significant within a district's mathematics program. With an awareness of just how important such an algebra selection process can be for students, the Westside Community Schools and the University of Nebraska at Omaha carefully examined Westside's algebra selection process by reviewing past placement data, holding a series of collaborative discussions, and then modifying the selection process to try to be as fair as possible to students within the context of limited district resources. This article describes an evidence-based investigation of Westside's algebra placement process and the related changes that the district made in its placement procedures as a result of this inquiry.

The Historical Context at Westside

First, it is important to get a sense of the Westside Community Schools. The district is an urban school district of approximately 6,000 students, 1,400 of whom are not residents of the district, but rather attend through Nebraska's school choice program. Eighty-six percent (86%) are white. Approximately 20% of the students qualify for free or reduced price lunch. The district has a K-12 curriculum with ten elementary schools (grades K-6), one middle school (grades 7-8), and one high school (grades 9-12). The district has always prided itself on having a strong and vibrant mathematics program, which has been recognized within the context of several awards, including students qualifying for the National Math Counts competition for five consecutive years, several students achieving perfect scores on the American Mathematics Competition and a high number of student qualifiers in the state's annual mathematics competitions.

During 2001, the Westside Community Schools adopted a new mathematics curriculum at the elementary level in order to better challenge their elementary students in mathematical problem solving as well as other higher level mathematics skills. The curriculum blends basic skills development with conceptual understanding activities in a mix that has been shown to be a positive component of effective mathematics instruction in several districts across the country (Cavanagh, 2006). The Westside program was carefully planned and adopted with considerable input from teachers, parents and even students (Grandgenett, Jackson, Willits, 2004). The elementary program revisions also included the adoption of *Everyday Mathematics* instructional materials, which appeared to align well with district desires to better challenge students. Elementary teachers also went through an extensive professional development program to help prepare them for a more challenging elementary curriculum. This professional development process also systematically included the early integration of algebra's big ideas, such as variables, patterns and functions, and proportions and proportional reasoning as recommended by authors such as Greenes (2004).

Teachers and students have embraced this revised elementary curriculum. Along with better preparing students for mathematical problem solving, reasoning, and mathematical connections, the curriculum also carefully covers introductory algebra topics which are well integrated into all grade levels at the elementary level. For example, in the Everyday Mathematics curriculum, algebra-related topics appear in each elementary grade and are indexed within the instructional materials (Everyday Learning Corporation, 2002).

Like most school districts today that have worked hard to develop an effective elementary mathematics program, placement into a formal algebra or pre-algebra course (leading to Algebra) at the middle school level has now surfaced at Westside as an important focus area for further revisions within the K-12 mathematics program. The district's strong elementary preparation in algebra readiness has only increased a need to offer strong middle school coursework options for students. Thus, the early integration of algebra concepts at the elementary level has essentially encouraged a more systematic approach to algebra at the middle school. This need for a careful transition for algebra instruction is consistent with research that suggests that successful instructional efforts for algebra should be well paced and systematic across the curriculum (Noddings, 2000; Steen, 1992).

In the National Research Council's 2005 report "How Students Learn," a total of 179 out of the 600 pages are dedicated to the learning of mathematics. Within this extensive discussion, Fuson, Kalchman, and Bransford (pgs. 217-256) reinforce that there are three important principles for teachers to follow in helping provide a foundation for the learning of mathematics, and particularly algebra. These principles include: 1) teachers must engage student prior understandings; 2) teachers must help students build a deep foundation of factual knowledge, give students a conceptual framework, and help them to organize knowledge; and 3) teachers need to help students take a metacognitive approach in taking control of their own learning within challenging coursework.

Challenging coursework has always been a strong component of Westside's mathematics program and student selection for such coursework has always been an important district concern. Historically, in the Westside district, two assessments were used to identify students who were perceived as "ready" for a challenging Pre-algebra course in the middle school after an aggressive elementary school curriculum. Students who received a score above the established cut scores were placed in Pre-algebra and others were placed in the "regular" 7th grade mathematics curriculum. This practice had a long history but no real documentation of the validity of the assessments or the predictive capability of the established cut scores. One of the primary assessments was even a "district-made" test that was initially constructed nearly 20 years ago by a group of middle school teachers and revised periodically over the years based upon the further input of later teachers.

The tests and the cut scores used for algebra placement had essentially not changed for more than a decade, but in recent years the proportion of students qualifying for Pre-algebra had steadily increased. The following table shows the percentage of students that took the placement tests each year and the percent qualifying within the district during the four years before changes were made in the selection process.

PRE-ALGEBRA TESTING

Year	6th Grade Enrollment	Number Taking Test	Percent Taking Test	Number Qualifying	Percent Qualifying
2001-2002	405	250	61.7%	137	33.8%
2002-2003	422	384	91.0%	248	58.8%
2003-2004	468	420	89.7%	283	60.5%
2004-2005	452	390	86.3%	258	57.1%

Although the tests and qualifying scores hadn't changed generally between 2001 and 2005 other things had. Historically, letters were sent to parents of students identified by sixth grade teachers as potential candidates for Pre-algebra. These parents were invited to have their child take the screening tests at the middle school on a Saturday morning or designated weekday evening, a practice that was eventually found to penalize students whose parents were not aware of, or initially interested in, providing this opportunity for their children. Procedures were then changed in the spring of 2002. Middle school teachers and counselors continued to administer the tests, but the tests were given during the school day at each elementary school and all students were encouraged to take the tests. As mentioned previously, the elementary curriculum had also changed during this period. The new curriculum placed greater emphasis on problem solving, reasoning, mathematical connections and had students apply their mathematical understanding to a greater extent than the previous curriculum. The curriculum also systematically introduced the "big ideas" of algebra at the lower grade levels. Standardized test scores in mathematics went up after the adoption of the new curriculum and teachers believed that the new curriculum also may have positively impacted students' performance on the Pre-algebra screening test.

As the numbers of students placed in Pre-algebra increased, middle school teachers recognized that the students arriving in these classes were representing a wider range of backgrounds and also observed that some students within this increased pool of students appeared to be struggling more than in the past. Two additional concerns led administrators to the conclusion that the placement tests and cut scores needed to be carefully examined. First, the validity of the tests themselves was in question. One test was a basic teacher-developed computational mathematics test, which had been refined over time, but without any formal reliability and validity testing. The other test was the Orleans Hanna, a commercially published assessment of algebra readiness (Harcourt Brace and Company, 1998). However, this more established test was not being used in connection with student grades as the test publisher prescribed. Secondly, there was no documentation of the formal procedures used to set passing scores on either of the assessments. There essentially was no evidence that the tests, or the established cut scores, were effective predictors of student success in Pre-algebra. Thus, the district felt it was time to carefully examine and better formalize the algebra placement process.

Looking at the Situation Statistically

To look at the algebra placement situation statistically and to better examine the algebra placement process, Westside partnered with the University of Nebraska at Omaha, to review the existing data related to the district's seventh grade mathematics placement process and compare the statistical power of the historical cutoff procedure with an alternate procedure thought to be more consistent with the new mathematics program. These two contrasting selection procedures included 1) the current use of the district constructed mathematics survey test (called the Westside Survey Test) and the commercially prepared *Orleans Hanna Test*, and 2) a potential alternate procedure using student grades and the *Orleans Hanna Test*. The alternate procedure using grades in combination with *Orleans-Hanna* scores, was also an assessment strategy recommended by the publisher of the *Orleans-Hanna Test*. In this context, grades were changed to a numerical score (again following *Orleans-Hanna*), using a scale of 0-12 for each grade assigned from F (assigned 0 points) to A+ (assigned 12 points). A total of 373 past student records were available to help investigate the relative statistical power of these two procedures.

As a first step in the statistical investigation, correlations were conducted to examine the overall relationships of various fifth grade and sixth grade mathematics variables (e.g., scores on mathematics assessments administered in fifth or sixth grade) with seventh grade mathematics achievement as represented by grades (see table below). The district also had a practical desire to have the qualifying procedure include a written test to aid in parent discussions. Another desire by the district was to somewhat emphasize the 6th grade scores since these scores would be more closely associated in time to the seventh grade year.

SAMPLE CORRELATIONS (6th GRADE)	r
Total 6th Grade Score	0.62
Mathematics Grade	0.60
Reading Grade	0.56
* Grades and Orleans Hanna Test Combined	**0.55**
Social Studies Grade	0.53
* Survey Test and Orleans Hanna Test Combined	**0.43**
Survey Mathematics Test	0.42
Orleans Hanna Raw Score	0.40
Science Grade	0.37
SAMPLE CORRELATIONS (5th GRADE)	**r**
Gr 5 SAT9 Total (Complete) Battery	0.45
Gr 5 SAT9 Total Math	0.42
Gr 5 SAT9 Math Proc	0.40

In examining the correlations, it appeared that the potential alternate selection procedure of combining semester "report card grades" with the *Orleans Hanna Test* was a viable alternative to the earlier procedure.

Multiple regression procedures were then used to compare the relative strengths of the two data models: the new model (Grades + *Orleans Hanna*) with the old model (Survey Test + *Orleans Hanna*) in their predictive relationships to student grades in seventh grade mathematics. The new model of combining grades and the *Orleans Hanna* scores was found to be statistically stronger when considering its effectiveness for achievement predictions within the available sample of 373 past student records. The new model accounted for 38% of the variance in scores, approximately double that of the old model, which accounted for only 19% of the variance. Actually, these findings are quite consistent with research that suggests that combinations of coursework grades and testing can be useful in predicting future mathematics performance (Burris, Heubert, Levin, 2004; Fenton, 2002).

Again using the historical data, the relative effectiveness of the two cutoff score strategies were then examined by considering how many "true predictions" and "false positives" the different cutoff score procedures represented while looking at the historical distribution of the 373 scores. For purposes of this comparison process, the following operational definitions were used:

> ***True Prediction:*** This term referred to the situation where a student made the cutoff score and then was successful in seventh grade mathematics.
>
> ***False Positive:*** This term referred to the situation where a student made the cutoff score, but was then unsuccessful in seventh grade math.
>
> ***Successful in seventh grade Math:*** A student was considered to be successful in seventh grade math if they received a grade of at least a "B" in their seventh grade math course.

As mentioned earlier, the current cutoff score procedure used a combination of tests that included the *Orleans Hanna Test* and a district created mathematics survey test. This traditional cutoff score process included the following criteria identified in district communications to parents:

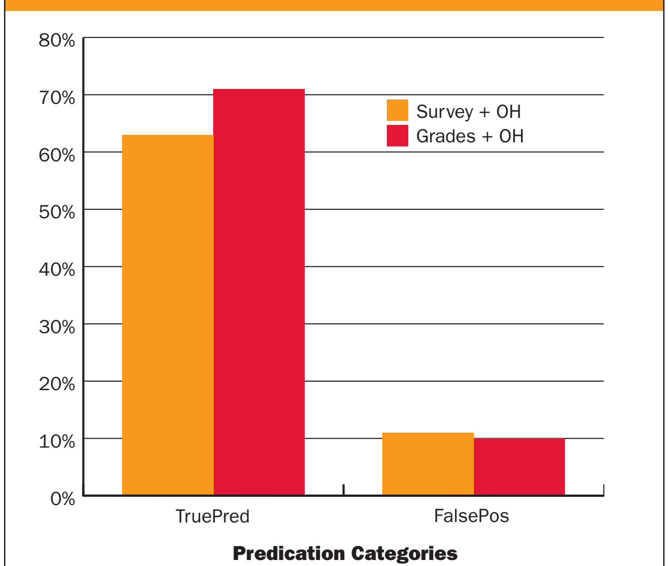
> "Students who are recommended for enrollment in the Pre-algebra course demonstrate the knowledge to be successful in Pre-algebra by meeting one of two criteria: 1) a score of 60% or higher on the Orleans-Hanna Algebra Prognosis Test and a score of 70% or higher on the Westside Mathematics Survey Test or 2) a combined average score on the two tests of 67% or higher."

This traditional cutoff score procedure predicted 63% of the sample's mathematics achievement (true prediction). About 11% of the sample were false positives (student made cutoff score but then struggled). It was also found by examining the 373 records that the two options within the criteria for qualifying (meeting the cut score on both tests or the mean of the two) statistically overlapped and were not both needed. All students either met both criteria or neither.

The recommended new student selection model used the *Orleans Hanna Test* and student grades. This selection process included a procedure recommended by the test publisher for combining student grades in four subjects (Math, Science, Social Studies, Reading/Writing). This approach uses the scale of 0-12 for each grade assigned from F (0 points) to A+ (12 points), and when combining all four grades, this point summation then accounts for a total grade value ranging from 0 to 48. This grade value is then combined with the *Orleans Hanna Test* scale of 0-50, to give an overall combined score ranging from 0 to 98. When examining the historical data, the new cutoff score procedure was found to be potentially superior based on this past data and a cutoff score of 64 was considered to be statistically optimum. Using this cutoff score, the prediction of student success (true prediction) was generally maximized and the false positives were relatively minimized (student makes cutoff score but is unsuccessful). This cutoff score predicted 71% of the population successfully, with 10% false positives.

Based on this analysis, the new cutoff score process was expected to statistically increase the true prediction of student success by roughly 8% while also potentially decreasing the false positives (student makes cutoff score but then

SURVEY TEST + ORLEANS HANNAH PREDICTIONS VS. GRADES + ORLEANS HANNAH PREDICTIONS



struggles) by roughly 1%. These two approaches are compared side by side on the graph.

Using the historical sample of 373 students to "predict" how many students would be expected to make the new cutoff score, it was determined that the new cutoff score process would most likely have about 67% percent of the district's students expected to quality for the initial middle school algebra course.

In essence, by using the new assessment procedure (combining student grades and the Orleans Test) it was concluded that there would be a more effective assessment process than the current procedure (using the Westside Survey Test and Orleans Hanna). The analysis of the historical data suggested that the new procedure would be more accurate, have slightly less of a chance of admitting students who would then struggle and would admit a few more students into the program. This new procedure would also make use of a test with greater demonstrated reliability and validity than a district constructed test.

The New System in Action

As expected, the new selection procedure resulted in nearly 67% of the students qualifying for Pre-algebra and has made the selection process easier to administer. Adding students' grades to the selection process using the numerical assignments as recommended by the *Orleans Hanna Test* is continuing to be monitored. Including grades and assigning the overall grade score to have an equal weight to the test itself, resulted in 35 students qualifying for Pre-algebra that would not have on the basis of the test score alone and disqualified 9 students that would have qualified on the basis of the test alone. The performances of these students are now being carefully observed.

As one might expect, we are finding that more advanced middle school mathematics coursework has significant implications for the mathematics curriculum throughout the secondary years. Increasing the number of students taking Algebra as eighth graders has the direct effect of increasing the number of students in advanced level mathematics in high school. The student who takes Pre-algebra as a seventh grader typically goes through a secondary course sequence that concludes with Calculus as a senior. Currently approximately 25% of the district's seniors take Calculus, roughly the same percentage that took Pre-algebra as seventh graders. Beginning with the new selection process for Pre-algebra in the 7th grade (and then Algebra in the 8th grade) the number of Calculus students at the high school level will potentially double.

As the district continues to review and adjust its mathematics placement process, some particularly talented students may well eventually become potential candidates for Calculus III as seniors. Historically the district has paid tuition for such students to enroll in Calculus III at a local University, but this will not be of interest for large numbers of students since Calculus III is required for only a few university majors. AP Statistics is being added to the high school course offerings to provide another option, but almost certainly, as more students are placed into early advanced coursework, the demand for higher-level mathematics courses in high school will grow.

Teacher perceptions continue to be mixed with the initial implementation of the selection process. Some teachers are skeptical that a larger percentage of students are able to handle Algebra and would still prefer a cut score resulting in fewer students being placed into the Pre-algebra sequence. Fewer identified students would indeed mean fewer students placed in Pre-algebra who do not perform well. However, it would also increase the number of students in seventh grade "General Mathematics" who might have been more appropriately placed in Pre-algebra.

The larger number of Pre-algebra students has also resulted in a scheduling challenge at the Middle School. Rather than six sections of seventh grade Pre-algebra, as was the case prior to the new selection process there are currently 11 sections. This change brings staffing and staff development implications. Teachers who have previously taught only seventh grade mathematics must be prepared to teach more challenging courses.

Although the greater numbers of accelerated students have required significant changes in middle school scheduling and staffing, the change has been particularly positive for scheduling in one important respect. Having a traditionally small number of accelerated students resulted in that group of students also taking other core curriculum courses such as English, Science and Social Studies together. This traditional procedure had the unfortunate effect of tracking throughout the system. With a larger number of students, it has been possible to schedule those students in a way that they can be better integrated throughout the system, minimizing the tracking across the middle school curriculum.

Next Steps: Where Do We Go from Here?

The changes related to algebra placement have been significant, but they are only just beginning. We will continue following the effectiveness and practicality of this new selection process. As greater numbers of students are placed and complete the courses, the statistical analyses we will conduct should be able to provide a more complete picture of how the new placement process is working. Curriculum review and staff planning is ongoing. High school staff and administrators have been involved throughout the change process and are fully aware of the implications. As more accelerated students advance through the system, significant changes will need to occur at the high school level. The high school will likely need to add Calculus III and certainly more sections of advanced mathematics classes will be needed. Who will teach these advanced level classes? That discussion is currently underway. Teachers who have taught Algebra and Geometry in the past will undoubtedly be asked to also teach these higher-level mathematic courses.

Finally, it is important that we continue the philosophical debate. There are those district educators who believe that only a very select group of students should be accelerated or take more advanced mathematics coursework. While at the other extreme, some educators believe that all seventh grade students should take Pre-algebra and that there should be no placement tests at all. We see such debate within the district as healthy and an important key to providing the best and most appropriate mathematics program for all students. Although we are still evolving toward a truly equitable and effective algebra placement strategy, we believe that we have made an important step forward with this revised and more inclusive placement process. As suggested by the John Kennedy, we also believe that "all children have an equal right to develop their talent, their ability and their motivation." Hopefully, the students in the Westside Public Schools are a step closer to realizing this important right with our mathematics curriculum.

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The Professional Development of Leaders and Teachers of Mathematics

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In 2003 with funding from the National Science Foundation (NSF/EHR-0314692), *Focus on Mathematics* (FOM) began its 5-year project to study, design and implement solutions to the problem of the large number of secondary school students experiencing difficulty with the learning and mastery of key ideas of mathematics. FOM is a collaboration of five Boston area school districts, their middle and high school mathematics leaders, teachers and students, and four institutions of higher education with their faculty in mathematics and mathematics education.

One of the first FOM activities was the establishment of the Curriculum Review Committee (CRC) composed of the math coordinators from the school districts and two Boston University faculty, one in mathematics and one in mathematics education. The initial charge for the committee was to review the districts' course and program syllabi for grades 6 through 12 to ensure that their topics and goals were in concert with the Massachusetts Curriculum Framework for Mathematics) and the Massachusetts Comprehensive Assessment System (MCAS) tests (www.doe.mass.edu/mcas).

After reviewing the materials, it was clear to the CRC that all programs and courses were aligned with the framework and with the tests. What then could account for students' poor performance on MCAS items? Several hypotheses were offered by the committee. The CRC's exploration of the hypotheses resulted in what we believe to be a fruitful and new approach to the professional development of leaders and teachers of mathematics.

We share this story with you because it 1) describes the genesis of our professional development program, 2) offers insights into students' fragile grasp of big mathematical ideas and skills, and 3) suggests the need to reconsider existing models used to represent key mathematical concepts. Although we focused our work in algebra, other mathematical content areas or ideas could just as effectively serve as the centerpiece of the activities. First, we present our story. This is followed by the Steps of the PD Model.

OUR STORY

In January of 2004, the mathematics leaders from the five school districts involved in FOM and the mathematics and mathematics education faculty at Boston University met for the first time. At that meeting, we realized that in order to tackle the problem of low performing students, we needed greater understanding of students' difficulties with major mathematical concepts. To gain some insight into those difficulties, we decided to spend several meetings analyzing student performance on the most recent MCAS tests, and on algebra items, in particular. The choice of algebra seemed to be a good one since the vast majority of teachers of students in grades 8 through 12 teach algebra or teach courses that involve the application of algebraic concepts and skills. Because the concept of linearity is fundamental to the study of algebra and a major topic in introductory algebra, the committee narrowed its focus to analysis of the linearity items and grade 8 student performances on those items.

What knowledge of linearity is required by the state test?

As a committee we "unpacked" each linearity item. The unpacking involved identifying 1) the mathematical concepts and skills that students need to bring to bear to the solutions of the problems, the reasoning methods required to solve the problems, and the types of displays or formats to be interpreted, and 2) possible reasons for students' difficulties with the items. What was particularly interesting in the discussion of #2 was that we teacher veterans did not reach consensus on the reasons for student difficulties with the MCAS items.

Why did students have difficulty with items on the state tests?

To gain consensus, we decided that we needed more input from the students. We selected three of the released linearity MCAS items that had appeared on the previous year's test and developed a clinical interview around the selected items. District math coordinators conducted and taped their interviews of grade 8 students solving the problems and describing their thinking. From the middle range of achievers, we selected students who liked to talk!

The entire committee viewed and reviewed the interviews and compared the "real" difficulties with our speculations. Yes, we were on target about 60% of the time. Surprisingly, we were not correct the rest of the time. Student difficulties were often totally unsuspected. We decided that we needed more information about the difficulties that we observed and the ones that we didn't suspect.

How can we get more information about student difficulties?

There was no existing vehicle for probing the specific difficulties so we spent several months developing our own assessment tool, which we will refer to as the Mini-Assessment Tool or MAT. For each item we developed scoring directions that would take note of specific types of errors.

The MAT consists of seven items that can be administered in one class period (see Appendix A). To reflect the formats of the MCAS items, the MAT contains one essay item, three short-answer items, and three multiple-choice items. A brief description of the seven items follows.

The Essay Response Item: Given coordinates of a point and the equation of a line, students determine if the point is on the line and describe their decision-making process.

The Short Answer Items: 1) Given an equation of a line with a negative slope, students create a table of values (coordinates) of points on the line. 2) Given a graph of a line, students identify the slope of the line. 3) Given a distance-time graph, students identify the part (one of three) of the graph that represents the car moving slowest; the slope of another part of the graph; and the car's speed in that other part of the graph.

Multiple-Choice Items: 1) Given a table of (x, y) values representing points on a line, students identify one of four graphs that contains all points. 2) Given a table of values showing the number of cars sold each week, students identify one of four linear equations that represents the relationship between number of cars sold and number of weeks. 3) Given a linear equation that is not in slope-intercept form, students identify one of the five possibilities for the value of the y-intercept.

One week after students completed the MCAS tests, Grade 8 classroom teachers in the five districts administered the MAT to all of their students and scored the tests. District leaders compiled results by school and by district.

What new information did we gain from our mini-test?

Our committee met shortly after all tests were scored to analyze the performance of the more than 3000 grade 8 students who completed the MAT. Findings revealed that across districts, students have minimal understanding of two major topics: points on a line and slope. (As an example, Figure 1 on page 18 shows the Data Reporting Sheet for Problem 1.)

With regard to points on a line, many students didn't know or weren't sure that: 1) all points on a line have two coordinates, including the y-intercept, 2) coordinates of points on the graph of a line satisfy the equation for that line, and 3) coordinates of points on a line that are presented in tabular form satisfy the equation for the line, and when plotted, produce a graph of the line.

With regard to slope, students demonstrated difficulty determining if lines shown in the coordinate plane had positive or negative slopes. Particular difficulty was noted when lines with positive slopes were pictured in the third quadrant of the coordinate plane. Many students could

FIGURE 1

PROBLEM 1	TALLY	TOTAL
a) Number of students who responded: **Yes**		
No		
b) Number of students whose explanations are: 1) **Replace** ***x*** with 2 and ***y*** with –8. Check that the two expressions are equal or **Replace** ***x*** with 2. Compute 3***x*** – 14(3 (2) – 14). Check result with the ***y*** value of –8		
2) **Replacement Error:** Error in computation after replacement		
3) **Graph:** Identify 2 points on the line. Construct graph. Locate (2,–8) on graphed line.		
4) **Graph Error:** Error in graphing or computing.		
5) Number of students who gave incomplete or incorrect responses different from those above.		
6) Number of students who didn't respond.		

not compute slopes from tables of data, graphs of lines, or equations of lines. Almost all students had difficulty recognizing the relationship between slope and speed in a time versus distance graph.

What more can we learn from the students themselves?

To validate our suspicions about the nature of the errors on the MAT (we learned a valuable lesson after interviewing students on MCAS items and discovering that our hypotheses about their errors were not always on target), district leaders conducted taped interviews of grade 8 students solving two of the MAT problems; one to determine the slope of a line from its graph and the other to interpret slope in an application problem. The difficulties cited above were confirmed in the interviews.

What could be contributing to student difficulties?

After viewing the tapes, and re-examining performance results on the MAT, district leaders were convinced that student difficulties stemmed from the format of problems, the grade level of the students, or the language and terminology used in the problem statements. Each of these sources of difficulty was investigated, and in the order listed.

Problem Format: In the original item on the MAT in which the slope of a line had to be determined from its graph, one of the points on the line was labeled with its coordinates and the line intersected the y axis at y = 4. However, there were no grid lines shown, and although the axes had hash marks, the scales were not indicated. The committee believed that the lack of grid lines and the unmarked axes presented a new situation for the students, one for which they were unprepared. To check out this hypothesis, three forms of that problem were developed and administered to about 1000 grade 8 students. One of the forms was the original presentation of the problem, a second showed grid lines, and a third showed grid lines and scales on the axes. Scores were analyzed and no significant difference by format was found.

Appropriate Grade Level: To check out the hypothesis that the concept of slope is too abstract for grade 8 students and should be explored with expected mastery by older students, the MAT was administered to all grade8, 9 and 10 students in the five districts. Performance was not markedly different by grade level. Grade 9 and 10 students experienced the same difficulties.

It was during the time when we were checking the relationship of grade level to success with MAT items, that our committee was joined by two international visiting faculty, Dr. Kyung Yoon Chang from Seoul, Korea and Dr. David Ben Chaim from Haifa, Israel. These faculty were quite interested in translating our MAT into their respective languages and administering the test to students in their countries. Dr. Chang gave the algebra test to Grade 8 students and Dr. Ben Chaim administered the MAT to grade 9 students. Students in those countries performed much like our students; they had difficulty with the same items and made the same types of errors.

Familiarity with Language: To explore difficulties posed by language, the mathematical language used in the 2000 through 2005 MCAS tests was compared with terminology in the instructional programs used in the five districts. Results of the comparison showed that all vocabulary in MCAS appeared in the various curricula.

Do instructional programs make a difference?

For two of the MAT items, one that focuses on identifying slope of a line from its graph, and the other that requires identifying the relationship between slope and speed, performance data were disaggregated by instructional program. Five different grade 8 programs were examined. In four of the districts, all students had difficulty with the two items regardless of the type of instructional program. In the fifth district, there remains the question of whether the ability levels of the students or the type of program made the difference.

As each new hypothesis was tested and results offered no explanation for student difficulties, the frustration level of the committee increased. The suggestion was made to examine the instructional programs. Perhaps the cause of the difficulties could be found in the programs themselves.

How do instructional programs differ?

How do instructional materials used in grade 8 introduce, maintain, and enrich concepts and skills of linearity, particularly those with which students have difficulty? Special attention was given to the models employed. Although some programs introduced the concepts in real-life settings and others used mathematical contexts, the models and language were essentially the same. Major differences were observed in 1) the particular chapter or unit when the concepts are introduced; 2) the nature of the types of applications presented; and 3) the frequency with which the concepts are revisited and practiced. We noted that, in all instructional programs, fundamental concepts of linearity were introduced but they were not reviewed and practiced systematically in order to enhance understanding and recall.

Do mathematics programs in grades 5 through 7 prepare students for algebra?

Another suggestion was made that the middle school curricula be studied to determine if students are being well-prepared to study algebra in grades 8 or 9. We are now asking ourselves, 1) What constitutes good preparation for Algebra I? 2) Is there a well-articulated development of mathematical concepts and skills in the grades preceding Algebra I that lead to the formal study of algebra? 3) For those applications of linearity that appear in middle school programs (and perhaps upper elementary school levels, as well), are students and their teachers aware that these are applications of the concept? Are these problems

THE *FOCUS ON MATHEMATICS* PROFESSIONAL DEVELOPMENT MODEL

Step 1: Select a content area focus and grade level(s).
We selected algebra and then looked more closely at an aspect of algebra. You can choose any content area of interest. Be sure that the focus area is not too broad.

Step 2: Analyze student performance on state or other tests.
Unpack items. Identify concepts and skills assessed, as well as displays to be interpreted. Speculate about student difficulties.

Step 3: Conduct video-taped interviews of students solving problems on the tests.
View tapes together, analyze student performances, and compare observed errors with your speculations about student difficulties.

Step 4: Design a mini-assessment tool (MAT) to probe student difficulties.
The format of items should reflect types used in your state assessments. Construct a scoring form that will help test administrators/teachers identify particular errors.

Step 5: Administer your MAT to all students and score the tests.

Step 6: Analyze performance on the MAT.
Speculate about student difficulties.

Step 7: Interview students solving MAT items.
As a group, view and analyze the tapes. Confirm/negate speculations about student difficulties.

Step 8: Examine instructional programs.
In what ways do they attend to student difficulties? Are other instructional materials/activities needed? How do earlier grades treat these concepts, skills and displays? Is there a clear articulation across grades?

Step 9: Develop a plan for scaffolding instruction of key concepts across grades.

highlighted and developed in such a way as to help students gain understanding?

What have we learned as a committee?

Although our committee has met once or twice each month since 2004, we still don't have answers to all of our questions. However, as teacher educators, we've learned a great deal of mathematics by "unpacking" problems and thinking about the concepts, sub-concepts and skills required to solve problems. We've become smarter about how to interpret student performance in mathematics and particularly, performance on local and state tests. We've gained greater insight into what students know and are able to do mathematically by interviewing them, and as a team, listening to and analyzing their comments, their interview behaviors, and their written work. We've become better at speculating about the nature of student errors. And, we've been humbled by how much about the learning of mathematics we still don't know.

We not only intend to continue our work as a committee, but we also want to engage our middle and high school teachers in the same type of study that we have done. We believe that the series of steps that were generated by our curiosity and observations constitute a new model for the professional development of leaders and of teachers of mathematics.

A FINAL NOTE

We believe that our own professional development was enhanced by the variety of districts and mathematics educators that were involved. If you decide to use this model, we strongly recommend that you join with at least one other school district and include university faculty in mathematics and mathematics education on your committee. If you are interested in administering our algebra MAT and comparing results with our districts, please contact Carole Greenes at cgreenes@bu.edu.

APPENDIX A

Please show all work on these pages

1. (a) Is the point with coordinates (2, –8) on the graph of the line $y = 3x–14$?

   (b) How did you decide?

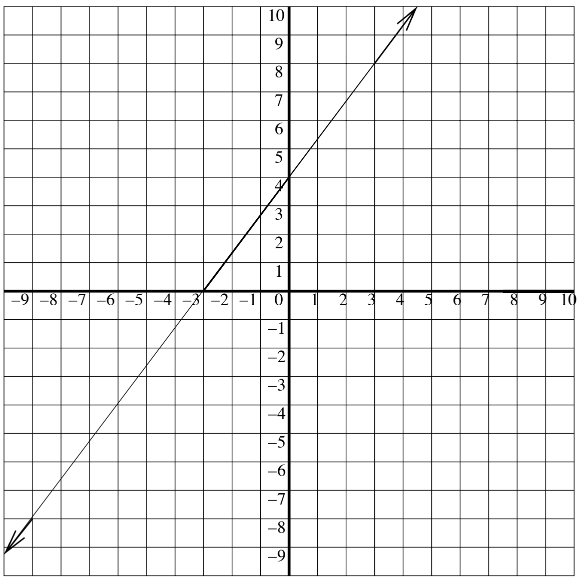
2. $y = 2x + 3$

   Create a table of values for the equation. Complete 5 rows of the table.

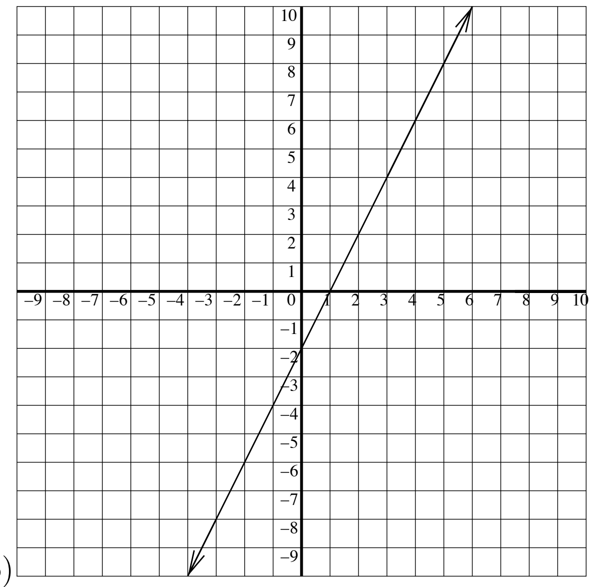
x	y
a.	
b.	
c.	
d.	
e.	

x	y
3	4
6	10
9	16
12	22

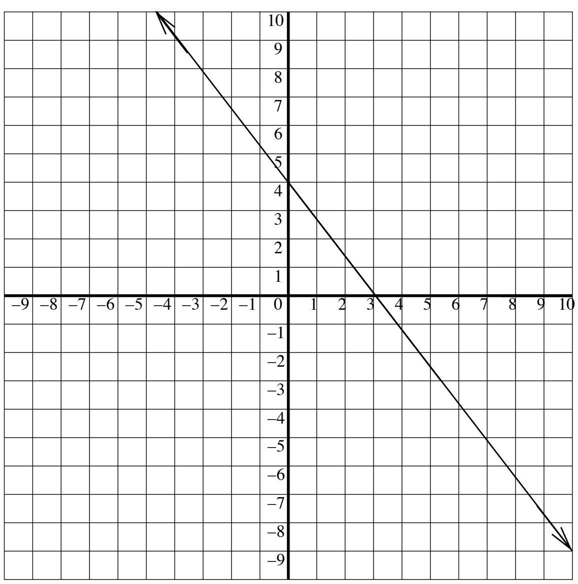
3. Which of the graphs below contains the points given in the table?



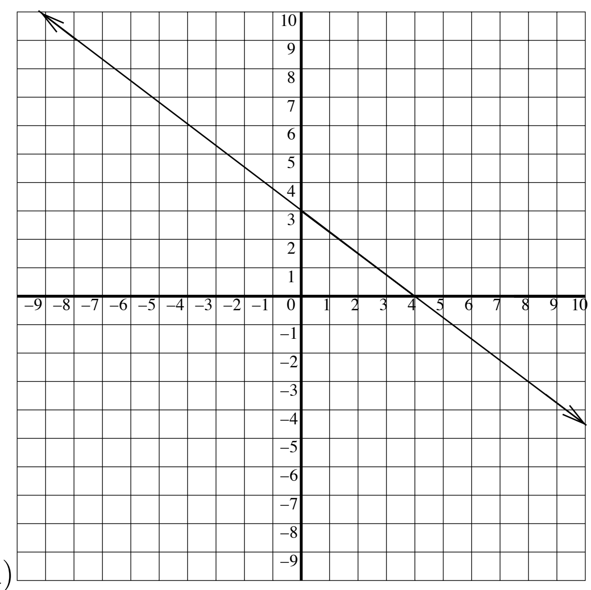
a)



b)

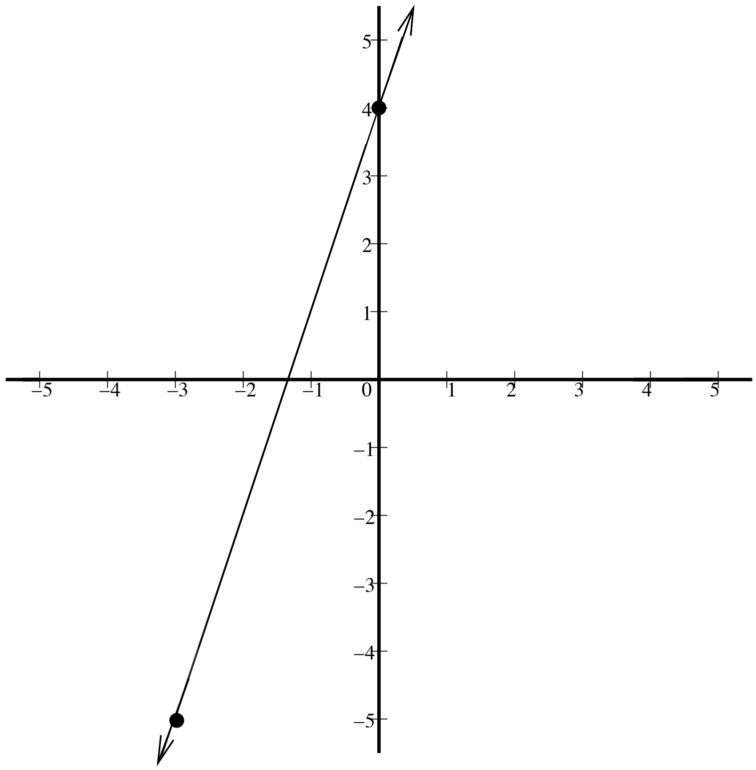


c)



d)

4. In the graph below, what is the slope of the line? Show your work.



5. The owner of a car dealership noticed a pattern in the weekly car sales, as shown in the table below.

Weekly Car Sales

Week *(w)*	**Number of Cars Sold** *(s)*
1	12
2	18
3	24
4	30

Which of the following equations represents the relationship between the number of cars sold (s) and the number of weeks *(w)*?

(a) $s = 6w$
(b) $s = 12w$
(c) $s = 6w + 6$
(d) $s = 6w + 12$

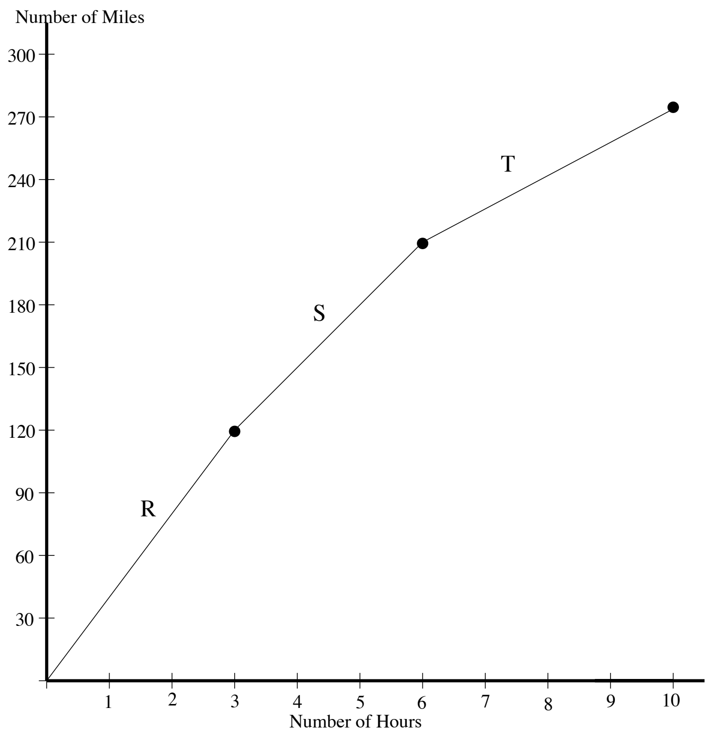
6. Given this linear equation

$$2x + 3y = 12,$$

which of the following is the y-intercept?

(a) 0
(b) 2
(c) 4
(d) 6
(e) 12

7. The graph below represents the distance that a car traveled after different numbers of hours



(a) Which part of the graph(R, S, or T) represents the hours when the car moved the slowest?

(b) What is the slope of part R of the graph?

(c) What is the speed of the car in part R?

The Interpersonal Side of Professional Development in Math

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What makes a successful professional development experience for elementary school math teachers? For over ten years, TERC's Investigations Workshops leaders have facilitated week-long professional development workshops across the country to support K-5 teachers implementing the *Investigations in Number, Data and Space* mathematics curriculum (Russell, Tierney, Mokros, & Economopoulos, 2006). *Investigations* emphasizes depth in mathematical thinking and reasoning, helping students develop flexibility in their approach to problem-solving, fluency in using mathematical skills and accuracy in evaluating solutions to problems. Engaging teachers in mathematics is a complex task. Many elementary school teachers learned math within a traditional curriculum — one that emphasized memorization and procedure over understanding (National Research Council, 2001). Consequently, elementary school teachers implementing *Investigations* and other programs that emphasize deep mathematical thinking are often uncomfortable with math or have gaps in their knowledge (Ball, Hill, & Bass, 2005). Moreover, some teachers mistrust the math content and pedagogy, doubting that changes will endure.

Consider the three-digit by one-digit multiplication problem 486 x 5. A traditional teacher might approach the problem by writing it this way:

$$\begin{array}{r} 486 \\ \times\ 5 \\ \hline \end{array}$$

The teacher may start out showing students to multiply the 5 by the 6, write the 0 and 'carry the 3' and then proceed to multiply 5 by 8 and by 4. We would like teachers to think of the problem in a variety of ways. For example, as 'half of 486 x 10' (or use estimation, noting that 4,860 is close to 5,000 so the answer should be just under half of 5,000; under 2,500) and solve it quickly, mentally, using both estimation and number sense. Or taking into account the value of 486, and multiplying as follows, 5 x 400 (2,000), 5 x 80 (400) and 5 x 6 (30), rather than breaking 486 up into three disconnected digits (4,8, and 6). To approach problems in these ways teachers need opportunities to solve them on their own, hear others' strategies, and develop a deeper understanding of the mathematics and an understanding of children's math thinking. This will enable them to question perspectives and strategies and facilitate children's learning as they guide them towards efficiency, accuracy and fluency.

Because the approaches that *Investigations* takes are often different from what's familiar to teachers, they may be apprehensive about attending professional development for *Investigations*. In addition to comfort with the math, teachers' attitudes and enthusiasm for attending professional development often depend on their involvement in the decision to select a curriculum for their district. As a result, some teachers arrive at workshops on guard, reluctant to attempt challenging math problems or eager to defend the math content and pedagogy that they find familiar and comfortable. So while Investigations Workshops focus on math content and pedagogy, leaders must tackle much more. To engage teachers, leaders must address all the dynamics that affect participation. Navigating this interpersonal dimension of professional development is among the most daunting and unpredictable aspects of facilitating sessions. To explore this issue we first review the literature on math professional development and then share three vignettes illustrating interpersonal challenges that arose at Investigations Workshops and how leaders addressed them. Each vignette raises ideas and questions with implications for professional development.

Recent research sheds light on what elementary math teachers need to know in order to convey math concepts to a diverse range of learners and how professional development can help build this knowledge. Mathematical knowledge for teaching goes beyond math content and computational accuracy. It also includes the "ability to unpack mathematical ideas, explain procedures, choose and use representations, or appraise unfamiliar mathematical claims and solutions" (Hill & Ball, 2004, p. 335). In addition to subject matter knowledge, this specialized teacher knowledge predicts student math achievement (Hill, Rowan, & Ball, 2005). These findings underscore the need for professional development intended to improve mathematical knowledge for teaching. For instance, in the multiplication example above, deep understanding of place value, estimation, and mental math is vital for understanding and supporting children's thinking. A growing body of literature highlights characteristics of effective professional development, such as an emphasis on math as embedded in the curriculum, strategies for teaching that math, and children's mathematical thinking (Loucks-Horsley, Hewson, Love, & Stiles, 1998; Cohen & Hill, 2001; National Research Council, 2001; Hill & Ball, 2004). Additionally, teachers in effective programs actively challenge their own and each other's thinking. Opportunities for teachers to reason, analyze, and communicate about math have been linked to gains in mathematical knowledge for teaching (Hill & Ball, 2004).

Modeling the constructivist pedagogy of the *Investigations* curriculum, our workshop leaders ask participants to solve problems and explain their reasoning in nontraditional ways, using manipulatives, representations, and mental strategies. In effect, leaders often ask teachers to confront their discomfort and challenge their reasoning. This can be scary, especially when participants find themselves in an unfamiliar setting, surrounded by new faces, as is often the case. Some participants must also adjust to a workshop that differs from their expectations. Many teachers arrive secure in their ability to solve and teach problems involving the four basic operations. They come anticipating an opportunity to add a few new ideas to their existing approach. Instead, by introducing unfamiliar constructivist pedagogies, we ask them to reconsider their whole foundation, a daunting task. Shulman (2000) notes, "When you begin to wrestle with people's deeply held, private intuitive theories, you are engaging them in a process that is as deeply emotional as it is cognitive. This is why conceptual change is so difficult to negotiate. When there is no pain, I suspect there has not been much conceptual change. The emotional aspect is something we have to learn to deal with" (p. 131).

Professional development leaders must understand the goals they and their participating teachers are working towards, and must recognize conditions that enable change. These are important steps. Yet many leaders still grapple with the delicate balance between forging ahead with the mathematics and pedagogy and attending to the emotional and interpersonal challenges that arise in professional development contexts (Miller, Moon, & Elko, 2000; Schifter & Lester, 2005). How can leaders promote norms of trust, respect, and active learning? How can they address resistance constructively? Teacher leaders know that the success of professional development sometimes hinges on these relationship variables.

During the past ten years, Investigations Workshops leaders have faced many interpersonal challenges and successes. Through debriefing sessions and conversations at annual leaders' retreats, leaders have identified and explored patterns in their experiences. Collectively, they have amassed a toolkit of strategies for addressing common challenges.

ALINA'S VIGNETTE: Engaging a Reluctant Participant

Alina's vignette highlights the importance and the challenge of accommodating individual differences in adult learners' participation styles.

It is the second day of a week-long workshop, and I am reading the "exit cards" that my participants completed as they left. One of the questions I posed was: What have you discovered about yourself as a learner? The card before me reads:

> *Did you notice that I sat in the back of the room with my arms crossed the first morning? Did you notice that I didn't talk at all in the large group until this afternoon? I was uncomfortable with the mathematics. And now we have completed our second day, and I found myself talking excitedly in my small group and I even shared my idea with the large group. Boy, that felt scary. Almost like*

jumping off the side of the pool into cold water. But now that I'm in, I have so much to say.

And I am thinking, *Yes, I did notice…*

In my role as a professional development facilitator, I need to consider both the goals of the professional experiences I offer and the needs of individual participants. A question that is always at the forefront is: How can I create an environment that supports the disequilibrium experienced during the construction of new understandings? This question demands consideration of how my facilitation supports and values the contributions of all teachers, even those who are reluctant to participate.

On day one, I greeted Shelly, who sat in the back. I gave her time and space to settle in. On the second day I grouped Shelly with people with whom I had seen her interacting and put the group at a middle table. When I shared some comments from the previous day's exit cards, I included one of hers. As I circulated during small group time the second day, I heard her share an idea with her group. I asked them to consider sharing that idea during the whole group discussion. Although Shelly didn't offer to speak, a member of her group began the sharing by saying, *"Shelly said…"*. In this way, Shelly's idea was made public. As the week progressed, Shelly became more animated and engaged. She stayed after our session on the fourth day to discuss a mathematical idea with which she was struggling.

Each new workshop brings a group of learners with a range of experiences in mathematics content and pedagogy. It is crucial that my first interactions with participants allow multiple entry points for connecting with our work. I must also establish an inclusive rapport and express genuine interest in their needs and ideas. Realizing that this type of learning environment must be carefully orchestrated, I:

- greet each participant before we begin each session
- begin sessions with time to reflect on prior learning and personal goals
- learn names and use them throughout the sessions
- ask participants to share concerns and questions on exit cards each day
- acknowledge and address those concerns as soon as possible
- model equity and respect
- maintain a brisk pace that also allows for adequate wait time and reflection
- discuss group norms and post them prominently
- listen carefully to participants' ideas in both small and whole group discussions
- honor the group's valuable time by focusing on their learning
- avoid external affirmation and foster intrinsic motivation through interesting and challenging tasks
- share my enthusiasm for learning as well as my interest in their ideas

On that first day, I could have interpreted Shelly's demeanor as disrespectful and her disengagement as confrontational. But to ensure that all participants engage with workshop goals, I must find a way to connect their needs, identify where they are, and offer them a way to enter our work. This is my responsibility as their facilitator. Once I offer them a safe place to try out ideas, opportunities to push their thinking, and authentic interest in their ideas, I begin to see the development of a learning community that values rigorous thinking and is willing and ready to pursue some common goals.

ALINA'S VIGNETTE:
Discussion

Contemporary educators recognize the value of personalizing schools to ensure that every child feels safe, welcome, and heard. Alina's experience with Shelly underscores the need to also personalize adult learning environments. Effective leaders "know that principals and teachers will only be mobilized by caring and respect, by talented people working together, and by developing shared expertise" (Fullan, 2001, p. 63). Nurturing the dynamics that enable strong learning communities to emerge and thrive is, for many, one of the most difficult leadership challenges. Since every group is different, there are no easy recipes for building community and fostering active learning. We can't always pluck the strategies used in one setting and apply them in another. Often, however, the experiences we have in one setting provoke questions relevant to other contexts. Alina's experience raises recurring leadership questions, such as:

- How can leaders accommodate diverse participation styles?
- How can we identify and respond to individual participants' concerns?
- How can we help each participant to feel recognized and valued?

There are times when even the most seasoned leaders walk away from a leadership experience feeling that they didn't build an effective learning community. The group didn't gel. Some participants seemed disengaged. The leader wonders if she inadvertently offended someone, undermining the supportive rapport she was working so hard to maintain. We often dwell on these disappointments, grappling with the tough questions they raise. This is important work, but revisiting positive experiences is worthwhile too as it enables us to wrestle with important leadership questions unencumbered by the emotional baggage of a disappointing experience. Positive experiences also give us hope, motivating our work and our efforts to improve. Alina's experience drawing initially reluctant participants into a learning community illustrates the assertion that professional development "can also be a vehicle for strengthening culture" (Loucks-Horsley et al., 1998, p. 185).

SAMANTHA'S VIGNETTE: Encouraging Open Minds

Samantha's vignette illustrates one way to address resistance to change, by acknowledging participants' stances first and then facilitating activities that challenge their thinking. In discussing factors that influence people's willingness to learn and change, Stone, Patton, and Heen (1999) write, "[People] are more likely to change if they think we understand them and if they feel heard and respected. They are more likely to change if they feel free not to" (p. 138).

We were at a workshop the summer after the participating teachers had struggled for one year to implement a curriculum that they did not fully accept or understand. This was their first professional development opportunity in support of the implementation. Many teachers were upset; the most outspoken earned a reputation in the district and at the workshop as the "vocal" group. I worked with the fifth grade teachers, who I had been warned were very angry. I know from working with other groups that participants really need to understand that I am there to support their progress.

I started the first session by addressing their feelings. As a former fifth grade teacher, I agreed that they had a tough job; teaching fifth grade math involves working through challenging mathematics concepts. I also acknowledged that their students had the least exposure to the math program, having had no opportunity to build the foundation established in the earlier grades. In short, these teachers experienced a trying year and I empathized with their frustration. I assured them that I would do my best to clarify the curriculum's content and pedagogy, but that I needed them to remain open to the ideas at least until day three. After that I would address all their concerns.

In essence I asked them to have faith in me for three days. If they didn't see a reason to give the workshop ideas a chance we could talk again on day three. As I spoke, I watched some heads nod and some participants' arms unfold and relax — small, but meaningful, signs that they were with me, and our work could begin.

The three-day agreement was a gamble because it often takes a fourth day for the workshop ideas to come together. In this case, day three came and went without incident. Teachers explored challenging math content, asked thoughtful questions, and worked together to build their understanding. By the end of the week, the fifth grade teachers, initially closed and angry, were willing to try *Investigations*. In their exit cards, the "vocal group" even expressed plans to listen to students' mathematical thinking and ask probing questions. They no longer dreaded the curriculum.

SAMANTHA'S VIGNETTE: Discussion

Ideally, professional development can prepare and energize teachers for change. But what happens when professional development comes in the wake of an unpopular change—one in which teachers had no say and no professional development support? Such conditions are often a recipe for anger. After all, teachers should have a voice in administrative decisions. They *should* have the support they need to implement new curriculum. In acknowledging teachers' frustrations, while preserving workshop goals and structure, Samantha displayed "tough empathy" (Fullan, 2001, p. 63). She also pulled off a difficult balancing act. She had to consider how to acknowledge participants' frustration, without allowing an angry mood to cloud her sessions and interfere with learning.

Additionally, by promising to revisit teachers' concerns if, by day three, they still did not find value in the curriculum, Samantha restored a critical component of the group's security — participants' sense of control. As leaders, we cannot change participants; it's our job to set the stage, facilitate learning, and help participants take control of the process.

While, in this case, Samantha's frank discussion sparked the group's willingness to learn, it is important to consider the risks and challenges of her approach. At what point does a discussion of frustrations become counterproductive? After acknowledging angry feelings, how can leaders help a group transition into constructive activities? If a leader requests that participants keep open minds for a few days, is the leader prepared to address a potential onslaught of concerns on the designated day? There are multiple ways to recognize feelings and invite feedback. The challenge is to do so without abandoning learning goals.

Exit cards, distributed in between sessions, provide one alternative strategy for soliciting feedback. Participants sometimes write comments on exit cards, such as "I like it much better when I show students how to solve a problem. This way I don't have to know all the ways in which a problem can be solved." or "This may work with the students in (the video), but my students…." These exit card comments illustrate participants' misgivings about making change and they provide leaders with an opportunity to address concerns that teachers might not raise in discussions. In addition to verbally acknowledging exit card comments, leaders can set up activities that allow teachers to explore the very issues that they are most concerned about. For example, teachers who initially believed that there is one 'best' way to solve a problem may begin to shift when they hear their peers solve 486 x 5 in a variety of efficient ways, as described earlier. Likewise, examining a diverse array of real elementary students' work and identifying learning goals to build each student's understanding, helps teachers begin to understand how they can meet the needs of the range of learners in their classrooms.

JASMINE'S VIGNETTE: Respectful Language

Jasmine's vignette emphasizes the importance of addressing equity issues in a session.

As a professional development leader, and as an African American female, I am aware that equity and respect issues emerge in professional development settings and need to be handled thoughtfully. These situations are difficult. The leader must keep emotions in check and avoid judgmental or defensive responses.

While facilitating the final session of a week-long workshop I found myself in a difficult situation. From day one we had established an open and safe environment. Participants felt free to express concerns, and I did my best to acknowledge and address their concerns. I was ending the final session when a participant commented on rubrics and scores. She referred to students who scored 1s as "the lows and the slows." Her tone suggested it was a routine phrase or possibly a joke among some teachers. I was appalled to hear students referred to so derisively and I assumed others in the room shared my discomfort. (She was an African American teacher and I had a strong feeling that she works with predominantly African American students.)

I had to think about how to respond without making it seem like I was attacking her. Finally I said, "I am concerned about the comment referring to students as the lows and the slows. We need to be careful about what we say about children, even to each other. You would be surprised at how many times the students hear what we say and that our remarks can have a long lasting effect on them." My comment seemed to be received okay — people seemed to listen and nod in agreement — and I hoped that using a "we" statement and not "you" deflected some of the judgment in my response.

I ended the session feeling that the last few minutes had gone in a direction that I did not anticipate. It could easily have turned into a nightmare. It was the end of the week. We were all tired. I could have let the comment slide, or worse, I could have botched the response with a harsh remark expressing my disgust. Throughout the week, in each session, we stressed that all children can learn math and talked about how to meet the needs of the range of learners in our classrooms. We used student work to assess each child's mathematical understanding and discussed our next steps as teachers. How then could a participant use assessment to label students in such a derogatory manner? I was astounded.

Having participated in discussions with fellow leaders about handling difficult issues, I was able to think on my feet and respond to the comment in a way that respected the speaker, the group, and the children who we were all there for.

JASMINE'S VIGNETTE: Discussion

Jasmine responded to a troubling comment without alienating the speaker or undermining the group's unity. She recognized her responsibility to model respect and she paused to carefully select an appropriate response. To a leader, those silences can feel eternal, but in enabling reflection and careful word choice, a long pause can mean the difference between havoc and harmony.

Teachers know that when an unexpected or tense incident occurs in the classroom—a child teases a classmate, a visitor drops in, equipment crashes to the floor—all eyes turn to the teacher. Children gauge their teacher's reaction and her response informs their own. This phenomenon, social referencing, begins in infancy and endures throughout the lifespan (Schaffer, 1996). When a contentious comment is made at a workshop, participants are likely to glance at each other and, most of all, to study their leader. Just as teachers must maintain composure in their classrooms —perhaps fighting the urge to roll their eyes as yet another announcement airs over the loud speaker — professional development leaders must constantly model respect with adult learners. This responsibility raises questions for leaders. In addition to our language, are we aware of our facial expressions, body language, and tones of voice? What factors influence our ability to react — a group's diversity, time, experience, beliefs? How do our experiences, race and ethnicity, socioeconomic backgrounds, and values affect our reactions to what we interpret as insensitive remarks?

As leaders, we are always being observed and our actions — or inaction — may be mirrored by those around us. It is easier to let an insensitive comment slide, but addressing it is an imperative part of modeling respectful practice. We cannot expect teachers to focus on math when they are distracted by a remark that puts down children.

Jasmine's vignette also illustrates that building a respectful community is an ongoing process, not a finite task limited to first days or ice-breakers. As a group's time together draws to a close, particularly in cases where members have bonded, some people will experience anxiety or a sense of vulnerability (Tuckman & Jensen, 1977). In responding to any situation near the end of a workshop, leaders must be careful to respect the delicate emotions that participants may experience as they prepare to leave the group.

Conclusion

Standards-based mathematics is a hard sell because many teachers are skeptical, overworked, and reluctant to make changes in their mathematics practice. Further, some teachers are uncomfortable with the math. While these obstacles make professional development work difficult, they also underscore its importance. When we have an opportunity to impact the way teachers teach and think about math, it is vital that we do it right. As with all types of teaching, facilitating professional development is about content, but it's also about being in touch with the participants. To maximize the impact of professional development on mathematical knowledge for teaching, we must be ready to address dynamics that can distract from the mathematics. While the Investigations Workshops focus on mathematics content and pedagogy, our leaders know that they must also focus on the intra- and interpersonal factors that affect participation.

There are many ways to handle any situation. Each approach carries potential risks and benefits. The three vignettes illustrate situations that can impede or facilitate growth and learning, depending on how they are handled. Alina reached out to a reluctant participant by creating a learning environment that accommodated different participation styles. Samantha faced the challenge of empathizing with participants' frustrations without letting their anger interfere with learning. Jasmine modeled respect by responding to an insensitive comment, while maintaining a positive rapport with the group. Reviewing relevant literature, sharing experiences, and problem-solving together helps facilitators build a repertoire of leadership strategies, so that when situations arise, they are better prepared to select a constructive response. A successful professional development experience involves more than content. As the three vignettes show, to get to the math we must also attend to the needs, emotions, and comfort levels that affect teachers' enthusiasm for learning math content and pedagogy.

Those leading professional development workshops, like Alina, Samantha and Jasmine, benefit greatly from discussions around hard issues. Making time for those discussions to take place among leaders is critical. While each group is different and the issues that surface in any given session may vary, it is helpful to think in advance about how or whether to address potential challenges. Before a workshop, learn about the group you will be working with. Even if they are from your school or district, each group has its own idiosyncrasies and character. When starting a session, work to establish a culture of trust. Participants will be 'with you' if they realize you are trustworthy and that it's okay to take risks and make mistakes. During your sessions, connect with the individuals as you would with students in a classroom — find out what the 'quiet ones' are thinking and provide opportunities for them to talk in small groups or pairs. Encourage participants to share strategies even when they hesitate or say that their approaches may be 'wrong.' If an individual or group's strategy does in fact reflect a misconception, ask questions that enable participants to rethink their approaches and build a stronger understanding. Explicitly communicate session goals and follow through. Don't be afraid to tackle and raise difficult issues, particularly those related to equity — those conversations are not easy, but they are critical if we believe that all children (and adults) can learn math.

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What?, Wow!, and Hmm…: *Video Clips that Promote Discussion of Student Math Thinking*

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Professional development programs incorporate the use of artifacts to promote mathematics teacher learning about a variety of topics, such as discourse, problem solving, and technology. Like many others, we use video of classrooms specifically to promote teacher exploration of student thinking about mathematics. We find, however, that not all video segments are equally effective — some classroom video excerpts lead to more substantive discussions than others. In our experience, it is not always the excerpts that we, as researchers, find most interesting that end up being productive for teachers. For this reason, we decided to engage in a program of research designed to help us understand what it is about certain video excerpts that makes them stimulating for teachers.

Our work takes place in the context of video clubs in which groups of mathematics teachers watch and discuss excerpts of videos of their classrooms (Sherin, 2000). We often serve as the video club facilitator, and in that role videotape participants' classrooms and select video excerpts to bring to the meetings. Because video from all participants' classrooms is typically viewed, the video club environment provides an opportunity to view a wide range of classroom practices.

SELECTING VIDEO CLIPS FOR TEACHER LEARNING

Prior research has, to some extent, considered the issue of how to design video excerpts to promote teacher learning. Much of this research focuses on technical considerations, for example, the importance of the sound quality and video camera positioning in the classroom (Roschelle, 2000). Similarly, some researchers discuss the advantages and disadvantages of particular recording formats, and the implications for how the recordings can be used by teachers (Brophy, 2004). In addition, researchers such as Lampert (2001) and Goldman-Segall (1998) discuss the inherent subjectivity of videotaping, and explain that video is not simply an objective reproduction of an event, but one perspective (that of the videographer) of what took place.

Other researchers, in contrast, look at the context in which the video is made. For example, there is general consensus that for video to be useful for teachers it must be authentic, and not staged. Along the same lines, some argue that teachers learn best when the video is representative of teaching contexts similar to their own (Brophy, 2004). Similarly, some argue that video need not illustrate best practices to be valuable, but that video which illustrates dilemmas that teachers encounter can also be quite constructive for teachers (Lampert & Ball, 1998; Seago, 2004). In our work, we extend beyond these broad considerations of the substance of video excerpts. In particular, we look closely at specific features of video that serve to illuminate student mathematical thinking for teachers.

a video case with the video episode as its centerpiece and includes four basic elements: situating the work, doing mathematics, viewing and discussing video, and linking to

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practice. The module map below helps to illustrate the activity flow of the module's eight sessions (figure 1).

THE MAPLETON VIDEO CLUB

The data presented in this paper are drawn from a year-long video club with seven fourth and fifth-grade teachers at an urban elementary school we refer to as Mapleton. The teachers met once or twice a month for a total of 10 meetings across one school year. Typically, videos from one or two teachers' classrooms were shown at each meeting, so that each teacher had an opportunity to share video on at least two occasions.

Administrators at the school and in the district invited university researchers to organize and facilitate the Mapleton Video Club. The purpose of the video club was to provide teachers with an opportunity to investigate the mathematical thinking of students in their classrooms. To that end, researchers videotaped in a few teachers' classrooms each month, and from those lessons selected excerpts to share at the meetings. There were a few instances in which teachers suggested specific portions of the video to use in the video club, but more often those decisions were left to the researcher-facilitator. In all, 26 video clips, averaging five minutes each, were shown across the 10 meetings. The clips represented a range of mathematical topics, as well as different types of classroom activities. The video club meetings were videotaped and transcribed for later analysis.

Characterizing Video Clips and Discussions of Student Thinking

As a first step towards our goal of understanding the types of video clips that prompt discussions of student thinking we investigated key features of the video clips shown in the Mapleton Video Club and the corresponding teacher discussions of these clips. In particular, we identified three features of the video clips that allowed us to distinguish different ways that video portrays students' mathematical thinking. We also identified criteria for establishing whether the teachers' discussion of the clip was *more* or *less* productive.

We claim that three dimensions of video reveal important differences in the student thinking exhibited: (a) the extent to which a video clip provides *windows* into student thinking, (b) the *depth* of student mathematical thinking shown in the video, and (c) the *clarity* of the student thinking portrayed. *Windows* refers to the types of evidence of student thinking provided in a video, such as verbal statements, written work, and gestures. *Depth* refers to the extent to which students are exploring substantive, rather than superficial, mathematical ideas. Finally, *clarity* concerns the ease with which a viewer can understand the ideas students share.

To examine this claim, all 26 video clips from the Mapleton Video Club were coded independently by two researchers as high or low on each dimension.[1] Inter-rater reliability was 85%. The resulting coding of the clips revealed a range along all three dimensions. (See Table 1 on page 34.)

Second, we characterized whether the teachers had productive discussions of the student thinking portrayed in the video clips. To do so, we analyzed three dimensions of the teachers' conversation: (a) the degree to which teachers focus on understanding student thinking, (b) the extent to which teachers explore substantive mathematical ideas, and (c) the extent to which teachers are engaged in joint sense-making concerning the interactions shown in the video. Specifically, discussions in which teachers consistently considered student ideas as objects of inquiry, discussed rich mathematical ideas, and responded to and built on each others' comments were considered *more productive.* Discussions in which this was not the case were considered *less productive*.[2] To be clear, those segments of discussion coded as less productive were not necessarily unproductive discussions; in some, but not all, of these segments the teachers had worthwhile discussions of topics other than student thinking: ability grouping, general and specific features of the mathematics curriculum, and the district stance towards mathematics learning. We chose to analyze only whether the discussions were useful discussions of student math thinking, as that was the focus of our professional development sessions, and hence the intended purpose for each video clip.

[1] In previous analysis, the video clips were coded as low, medium, or high on each dimension, with very few "medium" codes resulting (less than 10% out of 78 ratings). For the purposes of this paper, the medium rating was removed and those clips were re-coded.

[2] Interestingly, teachers' discussions were generally either strong or weak across all three categories. For example, there were no cases in which teachers consistently discussed substantive mathematical issues, but the discussion consisted of isolated and disjoint comments. For more information, see Sherin, Linsenmeier, & van Es, 2006.

TABLE 1. *Criteria for Characterizing Video Clips of Student Mathematical Thinking*

		Low	High
Windows into Student Thinking	Is there evidence of student thinking in the video clip?	Little evidence of student thinking from any source (e.g., very few comments from students)	Detailed information from one or more sources (e.g., student narrates and provides written account of solution strategy)
Depth of Student Thinking	Are students exploring substantive mathematical ideas?	Task is routine for student; calls for memorization or recall on part of student (e.g., student applies known algorithm)	Student engages in math sense-making, works on task at conceptual level (e.g., student devises invented strategy)
Clarity of Student Thinking	How easy is it to understand the student thinking shown in the video?	Student thinking not transparent (e.g., "What is that student talking about?")	Student thinking transparent; viewer sense-making not called for or single interpretation obvious (e.g., "She gives a very clear explanation.")

IDENTIFYING TYPES OF VIDEO CLIPS

Looking across our coding of the video clips and the video club discussions, we identified several patterns in the ways that particular combinations of *windows, depth,* and *clarity* resulted in more or less productive discussions of student thinking among the teachers (see Appendix A). In particular, in what follows, we describe six types of video clips: three that we have found lead to more productive discussions and two that typically did not lead to productive discussions. We also describe one clip type that initially resulted in less productive discussions but, over time, was a valuable resource for promoting productive discussion in the video club. As shown in Table 2, each type of clip represents a unique combination of *windows, depth,* and *clarity* of student thinking.

In presenting the six clip types, we discuss not only the three dimensions of video discussed above (*windows, depth,* and *clarity*), but also explore the relationship between clip type and several factors. To be clear, a number of factors that we examined appear to have no relationship with clip type. For example, the video excerpts varied widely in length, with the shortest lasting less than two minutes and the longest lasting a total of nine minutes. There was no connection, however, between clip length and clip type; in other words, a longer clip did not

TABLE 2. *Types of Video Clips of Student Thinking*

Video Club Discussion Type	Video Clip Type	Description	Characteristics of Student Thinking *Windows*	*Depth*	*Clarity*
More Productive Discussion	What?	"What is going on?"	High	High	Low
	Wow!	"I never thought of that!"	High	High	High
	Hmm…	"There's something interesting here."	High	Low	Low
Productive Over Time	Blip	A short burst of depth	Low	High	Varied
Less Productive Discussion	So What?	"I get it, but what's the point?"	Varied	Low	High
	Huh?	"I'm confused, but what's the point?	Low	Low	Varied

necessarily allow for greater *windows* into student thinking, more *depth* of student thinking, or higher *clarity* of student ideas. In addition, the video excerpts we selected portrayed students working on a wide variety of mathematical topics, from equivalent ratios, to decimals, to area and perimeter, yet no topic was more likely to produce video clips of a particular type.

Productive Video Clips Of Student Thinking

We identified three clip types that consistently led to productive discussions of student thinking. The *What?, Wow!,* and *Hmm…* clips are all high in *windows* but vary in the combination of *depth* and *clarity* of student mathematical thinking that they portray.

The *What?* Clip

What? clips are high in *windows* and *depth*, but low in *clarity*; in other words, they provide evidence of what students are thinking and the thinking is mathematically substantive, but something about the students' ideas is unclear. At the end of a *What?* video clip, we often find ourselves asking "What?" just happened. In our experience, these clips prompt teachers to explore student thinking in an attempt to answer that question.

In one *What?* video clip, for example, the teacher asks a student to explain the reasoning he used to obtain an incorrect answer to the following problem: if 1 inch represents 50 miles, then 1/2 inch represents how many miles? The student has said that the answer is seventy-five, and persists in this belief even though he acknowledges that "half a pizza is smaller than one whole pizza," and even though he essentially calculates the correct answer of twenty-five in the course of his explanation of why seventy-five is the answer. The teachers in our video club found this combination of both correct and incorrect reasoning very intriguing and spent quite a bit of time teasing apart the different aspects of the students' answer in order to reach a conclusion about what he understood.

What? clips often, although not always, involve students who explicitly express some sort of confusion about the mathematics they are doing, or students who obtain the incorrect answer to a problem. Mathematical mistakes, particularly in the context of reasoning and problem-solving, seem to be ready fodder for exploration. Perhaps student mistakes are inherently more interesting because there are so many different ways for students to do or think something that is mathematically incorrect. In contrast, when students obtain a correct answer, teachers are more often able to — rightly or wrongly — mentally fill in the blanks in a student's solution and at least believe that they understand the student's work. To be clear, some *What?* clips do illustrate a student who has provided a correct answer. However, in these cases, the student does not articulate his or her work clearly, and his reasoning is not transparent. Such clips, therefore, can be confusing for teachers who view them.

Another important feature of the *What?* clips that we identified concerns the context in which students shared their ideas. Specifically, *What?* clips took place either during whole class discussion or during interactions in which individual students were presenting their ideas at the board. In both cases, the classroom teacher verbally interacted with the student, asking questions about his or her ideas or methods. This teacher-student interaction was likely an important factor in promoting both high windows and high depth in these clips.

The *Wow!* Clip[3]

Wow! clips, like *What?* clips, are high in both *windows* and *depth*. In contrast to *What?* clips, however, *Wow!s* are high in clarity. Thus, students in *Wow!* clips are engaged in in-depth problem-solving and reasoning, but the viewer is left with little confusion about what students are doing or saying. *Wow!* clips are thought-provoking, and lead to productive discussions of student thinking, not because they provide teachers with something to figure out, but because they provide teachers with new insights into how to think about the mathematics that is presented in the classroom; teachers understand what is going on, but there is something interesting about the student thinking anyway.

Wow! video clips come in two varieties, those that contain innovative student methods and those that contain student errors. In the innovative clips, students use correct, but non-standard, solution methods. The viewer is able to

[3] As can be see in Appendix A, of the five Wow! clips we identified, four resulted in *more productive* discussions, while the discussion of one clip was coded as *less productive.* We believe this was the case because the video clip viewed immediately prior to this one came from the same teacher's classroom and portrayed a similar part of the lesson. Thus the teachers had, in a sense, already discussed the student thinking portrayed in this video clip and had no new ideas to add.

understand the students' work, but is excited about the new ideas and wants to pursue them further. For example, in one of our video club meetings, a teacher began the discussion of a *Wow!* clip by saying, "I would have never thought of doing it that way!"

While *What?* clips are often based on student mistakes that are confusing in nature, *Wow!* clips may contain fairly easy to follow student mistakes. These student mistakes, while understandable, can lead teachers to attend to aspects of the mathematics that they might otherwise not have noticed. For example, in one video clip a student is attempting to calculate the area of a rectangular figure, but repeatedly confuses perimeter and area. In their discussion of the video, the teachers in our video club realized that this perimeter versus area distinction was one that they probably needed to consider and make more explicit than they had in the past.

Like *What?* clips, *Wow!* clips primarily drew from participant structures in which the classroom teacher plays a significant role, such as instances in which there were whole class discussions, in which students were presenting solutions at the board, or in which the teacher was talking with an individual student. As before, the teacher-student interaction was likely an important factor in promoting both high windows and high depth in these clips. Interestingly, one of the *Wow!* clips takes place in a student-to-student context, involving a pair of students working without their teacher. In the video club discussion of this clip, a Mapleton teacher comments that one of the students "was acting like she was the teacher there." It seems that, while *What?* and *Wow!* clips may require the active participation of a teacher, that role can occasionally be taken on by students.

The *Hmm…* Clip

Hmm… video clips are high in *windows*, but low in *depth* and *clarity*. In other words, the thinking portrayed is routine and algorithmic in nature, but despite the use of routine thinking, something about the students' ideas is unclear. *Hmm…* clips, like *What?* and *Wow!* video clips, are good prompts for productive discussions of student thinking.

In addition, we found that *Hmm…* clips always involve student mistakes and confusion. Although the student mistakes may be on a superficial level mathematically, the teachers talk about the mathematical concepts underlying these mistakes in order to understand them. In doing so, the teachers go beyond the mathematics in which the students in the video are engaged, and have a discussion that is mathematically substantive.

In one *Hmm…* clip, a pair of fourth-graders is practicing their single-digit multiplication facts in the context of a card game. The only talking in the video clip is short comments such as, "I got forty-eight," and "sixty-four, I guess I win." There is gestural evidence, however, of students counting to reach their answers. The students in the video are merely practicing multiplication, but teachers who watch the video clip are curious about why the students make certain mistakes. In particular, the teachers want to know whether the students actually understand the concept of multiplication, even though they often give incorrect answers. For the teachers, making decisions about what the students do and do not understand involves having a mathematically rich, very productive discussion about student thinking.

In our experience, *Hmm…* clips take place in the context of what we call the "student-to-student" participant structure, that is, in cases in which a group of students is working together without the significant presence of a teacher. When students work together there is often a lot talking and gesturing, leading a "student-to-student" video clip to be high in *windows*. However elementary school students, on their own, do not always effectively question each others' ideas or ask for further explanation. Thus, a video clip without the involvement of a teacher is more likely to be low in *depth* and *clarity*.

Productive Over Time

While most clip types appear to lead to only *more productive* or *less productive* discussions of student mathematical thinking, one type of clip, the *Blip*, actually became more productive over the course of the video club; of the five *Blips* shown in our video club, the first three led to less productive discussions of student thinking, whereas the final two led to more productive discussions. Perhaps the easiest way to understand the nature of a *Blip*, which is low in *windows*, high in *depth*, and varies in its degree of *clarity*, is to think of it as a "fleeting" *What?* or *Wow!* — for much of the clip, there may be no significant student ideas, but the clip contains short glimpses into what students are thinking. In these short bursts of depth, the student ideas may be clear or unclear, but the ideas themselves are thought-provoking.

In addition to being the only type of clip that prompted multiple productive, and multiple unproductive, discussions of student thinking, *Blips* are also the only type of clip with low windows that we have seen lead to productive discussions of student thinking. In fact, we believe it was precisely the low windows that lead to its variable effectiveness. Specifically, in related research, van Es (2004) explains that early in the Mapleton Video Club meetings, the teachers were not skilled at identifying key moments in the video that required closer attention, particularly moments in which interesting student thinking was visible. Thus, we infer that if students' ideas were represented in much of the clip, teachers were more likely to attend to these ideas; in contrast, if a student's idea was mentioned only briefly, it was not likely to gain the attention of the teacher. Over time, however, van Es found that the participants in the Mapleton Video Club developed a more refined ability to notice student thinking. Therefore, later in the series of video club meetings, it seems more likely that they could productively attend to the kinds of short bursts of deep student thinking that *Blip* video clips contain. In a sense, the "small windows" in Blip video clips are like the peepholes in apartment and hotel room doors; much of the time, nothing can be seen through them, but if one is looking in just the right way, quite a lot is revealed.

It is worth noting, also, that all instances of *Blip* video clips came from whole-class discussions. These were cases in which, in the midst of discussion, a student raised a substantive idea, sometimes making an insightful, correct comment and at other times making an mathematical error. In a *Blip* video clip, however, the idea is not pursued by the teacher. Instead, the teacher may simply correct any mistakes or acknowledge that a new idea has been raised, and then move on without further discussion.

Unproductive Video Clips of Student Thinking

We have found that two types of video clips consistently lead to less productive discussions of student thinking. It seems that a lack of mathematical *depth* in conjunction with certain degrees of *windows* or *clarity* can cause a video clip to be uninteresting for teachers. *So What?* clips combine low depth with high clarity, whereas *Huh?* clips combine low windows with low depth. Furthermore, in both cases, these clip types are represented by a variety of participant structures. This suggests that no particular classroom arrangement will guarantee a productive combination of *windows, depth,* and *clarity*.

The *So What?* Clip

So What? video clips vary in the degree of *windows* they contain, but are low in *depth* and high in *clarity*. These video clips lead to unproductive discussions of student math thinking because neither the mathematics nor the students' ideas themselves are thought-provoking. The students' ideas are clear, so, unlike with *What?* and *Hmm…* clips, there is no work to be done to understand what the students are thinking. In addition, the mathematics in *So What?* video clips is routine and based on rote-recall, so there is little motivation to explore the mathematical ideas that are raised in the video clip. A *So What?* clip is easy to understand, but is simply not very interesting.

The *Huh?* Clip

A *Huh?* video clip is low in *windows* and *depth*, but can vary in *clarity*. The combination of low windows and depth means that, even if student ideas are unclear, teachers may not feel that it is worth making the effort to figure out what students are thinking. Thus, discussions of the student thinking in *Huh?* clips tend to be unproductive. In contrast to a *Hmm…* clip, *Huh?* clips do not have sufficient windows to be used as a jumping off point for trying to understand confusing student ideas. Furthermore, in contrast to a *Blip*, in which the substantive mathematical ideas counterbalance the minimal evidence, the mathematics in a *Huh?* clip is not thought-provoking.

In one *Huh?* video clip, a group of students is filling out a worksheet that begins by telling them that a single sheet of paper contains 2,000 dots, and then asks how many dots would be on five pages, fifty pages, five hundred pages, and so on. The worksheet is essentially an exercise in correctly using place-value in numbers that are multiples of ten. The clip contains very little evidence of student ideas because the students only give partial explanations of their answers, and we cannot see what the students are writing on their worksheets. In their discussion of this video clip, the teachers in our video club spent a significant amount of time just trying to decide whether students were answering the worksheet questions correctly; while the teachers are attempting to make student ideas an object of inquiry, they cannot move on to interpret the meaning of student comments, or to think about the mathematics involved, unless they are able to first accurately identify student ideas. At one point in the discussion of this video clip, a teacher makes the telling comment, "Oh, who cares about the…dots anyway." This teacher is acknowledging her frustration that even if she were able to eventually understand the ideas contained in this video clip, they would not be of a significant nature — and thus worth the effort — anyway.

The *Double Whammy* Clip

Double Whammy video clips are not truly an additional type of clips, but are merely those video clips that fall into both the *So What?* and *Huh?* categories. In other words, *Double Whammy* clips are those that contain low windows, low depth, and high clarity of student thinking. We consider *So What?* and *Huh?* video clips to be separate types because they lead to unproductive discussions for different reasons. The existence of *Double Whammy* clips, however, may still be significant. It is our hypothesis that, while *So What?* and *Huh?* video clips might occasionally lead to productive discussions,[4] it would be particularly difficult to have a productive discussion about a *Double Whammy* video clip because such clips are "doubly" problematic.

DISCUSSION

When originally selecting the 26 video clips to be used in the Mapleton Video Club, we had neither the video clip dimensions (*windows, depth,* and *clarity* of student thinking) nor the video clip types in mind. Our goal had been to pick "good" clips where "something interesting was happening." In retrospect, our view of "interesting" student thinking was closest to the *What?* clip, this is excerpts that involved substantive, but confusing, thinking on the part of students. Our belief was that these clips would prompt teachers to want to explore and understand the student ideas portrayed.

We were not surprised to learn that *Wow!* clips also lead to productive discussions of student thinking. While the clarity of these clips meant that teachers might not be prompted to understand student ideas per se, we still expected teachers to be interested in the deep mathematical concepts underlying those ideas. The lesson of the *What?* and *Wow!* clips is that, at least in the context of sufficient evidence, teachers do consistently become engaged with the substantive mathematics in a video clip, regardless of the clarity with which students present their thinking.

In contrast, we were surprised to learn that teachers were able to have productive discussions of student thinking even when discussing a video clip in which students were not exploring mathematics in a substantive way. As researchers, we did not find the *Hmm…* clips nearly as engaging as the *What?* and *Wow!* video clips, but in the course of analyzing the Mapleton Video Club data, we discovered that, in the right context, routine and algorithmic mathematics can still provide a useful prompt for teachers.

Prior to conducting the analysis of the Mapleton Video Club data, we expected *Blip* video clips to lead to productive discussions far more consistently than they did. As part of our belief that high depth was a key component of "good" video clips, we thought that any clips that contained mathematical depth would be good. As researchers, we had become skilled at finding moments of mathematical depth, however fleeting they might be. As it turns out, the teachers in our video club did not have the needed experience to be able to focus on these shorter instances of depth. *Blips* were always interesting to us as mathematics education researchers, but teachers needed time and experience to see them.

Other clip types, namely the *So What?* and *Huh?* video clips, were unproductive as we had expected. The selection of these video clips arose from the constraints of running a video club, but the inclusion of such video clips allowed us to confirm our hypothesis that low depth in combination with either high clarity (the *So What?* clip) or low windows (the *Huh?* clip) will indeed make it difficult for teachers to have productive discussions of student thinking.

What, then, are the lessons that we can learn from our knowledge of these six video clip types? The first lesson is that we must be careful about using *Blips* in professional development. That is not to say that video clips containing only short bursts of student sense-making should never be shown, but that they should be saved until after teachers have honed their interpretation skills. The second lesson is that, while eliminating *Blips* from the collection of useful video clips — at least at first — may appear to limit the range of productive clip types, the addition of *Hmm…* clips also expands it. This knowledge gives teacher educa-

[4] Clearly, factors other than the video clip can affect the nature of the teachers' discussion, for example, who is present at any particular meeting, or the teachers' familiarity with the lesson that is viewed. In fact, in the first Mapleton Video Club meeting, the teachers have a productive discussion of student thinking in the context of a *Huh?* clip. We believe this was due to the strong direction provided by the facilitator in an attempt to establish norms of analysis.

tors additional flexibility in selecting video to use with teachers. One need not look for the "perfect" lesson; one need only look for a useful combination of *windows, depth,* and *clarity* to use at the right time.

Our interest in identifying types of video clips initially stemmed from our use of video in video clubs. In this context, we faced two constraints in selecting appropriate video clips: (a) we wished to show video from all participants' classrooms, yet student thinking was exhibited quite differently across teachers' instruction, and (b) in contrast to many professional development programs which have a long time span over which to produce the ideal video clip, we were required to choose clips once or twice a month from only a small number of observations. Without the flexibility to search through many hours of video in order to find the best excerpt, it was particularly important to us to be able to predict what kinds of video clips would lead to productive discussions of student thinking.

We believe this research will not only be useful in conducting future video clubs, but also has implications for teacher education and professional development more broadly. For those designing video-based professional development materials, it can be valuable to be aware of the different clip types that may lead to more or less productive discussions. Even when professional developers have time to search for video across many classrooms, they may find that this research provides a useful framework for focusing their attention.

Furthermore, in mathematics education in particular, a wide range of video-based materials are available. Teacher educators who select from among these materials often draw from multiple sources at different points in a course. The clip types presented here can serve as a guide to such instructors of what might be useful and why.

While the work presented here adds to our understanding of the role of video in teacher learning, additional questions remain. In the future, we hope to explore what teachers learn from viewing and discussing different types of clips. We suspect that viewing multiple kinds of clips will provide the most benefits for teachers as they develop their skills in interpreting student thinking in different contexts. We also wish to explore how to most effectively facilitate different types of clips. For example, understanding that in *Hmm…* clips the goal is to have teachers move beyond the mathematics that the students explore might influence the types of questions the facilitator poses to the group. Other research might also want to explore how video clips can promote productive discussions of topics other than student thinking. We suspect that the methods presented here can be adapted for such purposes.

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APPENDIX A

Six of Types of Video Clips Identified in the Mapleton Video Club

		Characteristics of Student Thinking WINDOWS	DEPTH	CLARITY	CODING OF VIDEO CLUB DISCUSSION
WHAT?		**HIGH**	**HIGH**	**LOW**	**MORE PRODUCTIVE**
Video Club 3	Clip B	High	High	Low	More Productive
Video Club 5	Clip A	High	High	Low	More Productive
Video Club 6	Clip B	High	High	Low	More Productive
Video Club 7	Clip A	High	High	Low	More Productive
Video Club 8	Clip D	High	High	Low	More Productive
Video Club 8	Clip E	High	High	Low	More Productive
Video Club 10	Clip D	High	High	Low	More Productive
WOW!		**HIGH**	**HIGH**	**HIGH**	**MORE PRODUCTIVE**
Video Club 2	Clip C	High	High	High	More Productive
Video Club 3	Clip A	High	High	High	More Productive
Video Club 7	Clip B	High	High	High	More Productive
Video Club 8	Clip C	High	High	High	More Productive
Video Club 9	Clip B	High	High	High	Less Productive
HMM...		**HIGH**	**LOW**	**LOW**	**MORE PRODUCTIVE**
Video Club 4	Clip A	High	Low	Low	More Productive
Video Club 10	Clip A	High	Low	Low	More Productive
BLIP		**LOW**	**HIGH**	**VARIED**	**VARIED**
Video Club 1	Clip A	Low	High	Low	Less Productive
Video Club 2	Clip B	Low	High	High	Less Productive
Video Club 8	Clip A	Low	High	Low	Less Productive
Video Club 9	Clip A	Low	High	High	More Productive
Video Club 9	Clip C	Low	High	Low	More Productive
SO WHAT?		**VARIED**	**LOW**	**HIGH**	**LESS PRODUCTIVE**
Video Club 2	Clip A[1]	Low	Low	High	Less Productive
Video Club 6	Clip A[1]	Low	Low	High	Less Productive
Video Club 8	Clip B	High	Low	High	Less Productive
Video Club 10	Clip B[1]	Low	Low	High	Less Productive
Video Club 10	Clip C[1]	Low	Low	High	Less Productive
HUH?		**LOW**	**LOW**	**VARIED**	**LESS PRODUCTIVE**
Video Club 1	Clip B	Low	Low	Low	More Productive
Video Club 2	Clip A[1]	Low	Low	High	Less Productive
Video Club 4	Clip B	Low	Low	Low	Less Productive
Video Club 6	Clip A[1]	Low	Low	High	Less Productive
Video Club 10	Clip B[1]	Low	Low	High	Less Productive
Video Club 10	Clip C[1]	Low	Low	High	Less Productive

[1] These video clips appear in both the *So What?* and *Huh?* categories. We refer to them as *Double Whammy* video clips.

The Problem-Solving Cycle:
A Model of Mathematics Professional Development

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ABSTRACT:
There is a growing consensus that mathematics teachers need to significantly expand their content and pedagogical content knowledge in order to make instructional improvements and provide increased opportunities for student learning. Long-term, sustainable professional development programs can play an important role in this regard. Our research team has spent the past several years developing a program called the Problem-Solving Cycle (PSC). This professional development model is grounded in a situative perspective on learning and draws upon theoretical and empirical evidence regarding the importance of professional learning communities and the value of using artifacts of practice to situate teachers' learning in their classroom experience. The model takes into account the complexity of classroom teaching, the wide array of knowledge teachers need to promote the mathematical thinking of their students, and the long-term commitment required to develop such knowledge. In this article, we present the conceptual framework for the PSC, details of its enactment, and initial findings regarding its impact on teachers' knowledge.

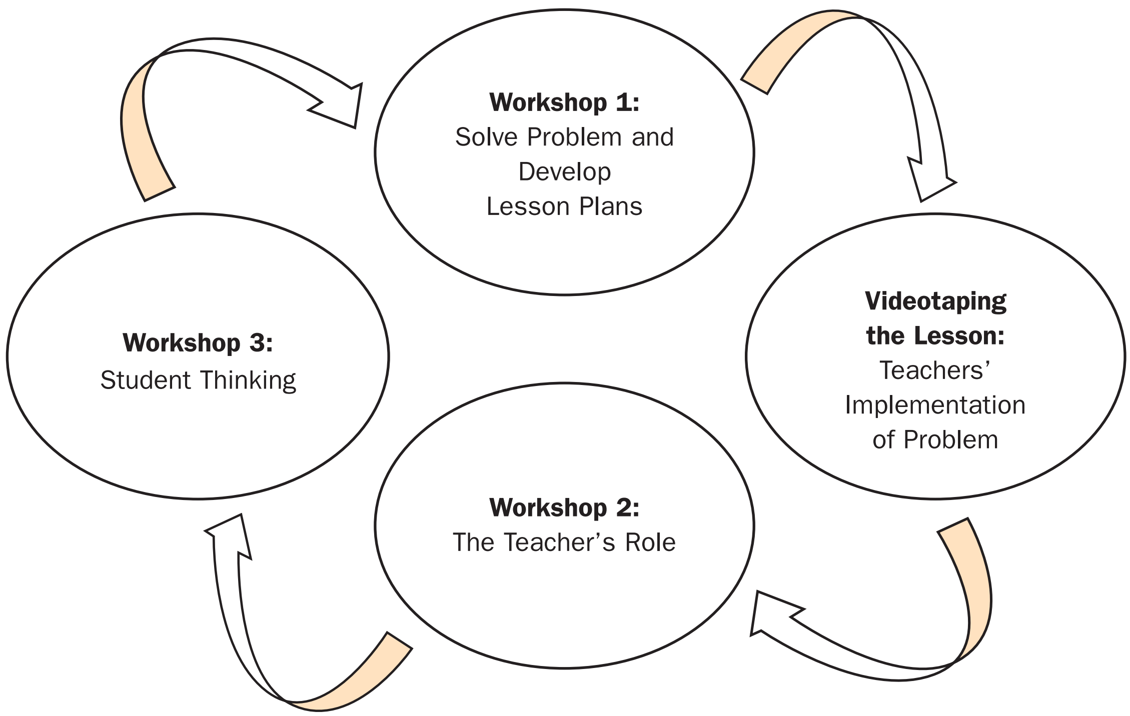
The Problem-Solving Cycle (PSC) model of mathematics professional development is an iterative, long-term approach to supporting teachers' learning. One iteration of the PSC consists of three interconnected workshops in which teachers share a common mathematical and pedagogical experience, organized around a rich mathematical task. This common experience provides a structure within which the teachers can build a supportive community that encourages reflection on mathematical understandings, student thinking, and instructional practices.

During the first workshop of the PSC, teachers collaboratively solve a rich mathematical task and develop plans for teaching it to their own students. Workshops two and three focus on teachers' experiences implementing the task in their classrooms (see Figure 1). The teachers consider more about the mathematical concepts and skills entailed in the task, their instructional strategies, and their students' mathematical thinking. In all three workshops, there is an emphasis on using artifacts of practice to situate teachers' learning opportunities in the context of their work.

One iteration of the PSC roughly corresponds to an academic semester, so that teachers can participate in 2 iterations (6 workshops) per school year. Each iteration focuses on a unique mathematical task and highlights different aspects of teachers' instructional practices and students' mathematical thinking. Successive iterations of the PSC build on one another and capitalize on teachers' expanding knowledge, interests, and sense of community. The PSC model is designed to be implemented by a knowledgeable facilitator, who carefully plans and conducts each workshop and continually monitors the participating teachers' needs and interests. The facilitator might be a teacher leader, mathematics coach, department chair, professional development specialist, or other teacher educator.

The PSC model is flexible with respect to the domain of mathematics that is selected as well as the specific learning goals and instructional strategies that are addressed. In our

FIGURE 1: *The Problem-Solving Cycle model of professional development*



Note: The arrow from Workshop 3 to Workshop 1 represents movement from one iteration of the PSC to the next.

work, we have focused on algebra because of the growing concern regarding students' inadequate understanding and preparation in this domain of K-12 mathematics (U.S. Department of Education and National Center for Educational Studies, 1998). Algebra operates as a "gate-keeper" to higher mathematics and future educational and employment opportunities (Ladson-Billings, 1998; NRC, 1998). Students' difficulties in learning formal algebra are well documented (Kieran, 1992; Nathan & Koedinger, 2000), and our schools' approaches to algebra instruction are lacking. For example, first-year algebra courses have been characterized as "an unmitigated disaster for most students" (NRC, 1998, p. 1).

The enhancement of teachers' professional knowledge about algebra and the teaching of algebra is considered to be a central component in the effort to support students' algebraic reasoning (Blanton & Kaput, 2005; Lacampagne, Blair, & Kaput, 1995; NCTM, 2000). The PSC model was developed and implemented as part of the Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR) project[1], which aimed to help teachers enhance their professional knowledge for the teaching of algebra and improve their instructional practices. We focused on middle school because it is becoming more common for school districts to require that algebra be taught during the middle school years, yet many middle school teachers have limited experience in teaching algebra. Furthermore, their experiences as algebra students typically emphasized learning procedures and manipulating symbols rather than reasoning about algebraic ideas (Ball, Lubienski, & Mewborn, 2001).

[1] The professional development program is one component of the STAAR Project, supported by NSF Proposal No. 0115609 through the Interagency Educational Research Initiative (IERI). The views shared in this article are ours, and do not necessarily represent those of IERI.

CONCEPTUAL FRAMEWORK FOR THE PROBLEM-SOLVING CYCLE

In this section, we present the conceptual and empirical grounding for the goals and processes of the Problem-Solving Cycle. We first explore the professional knowledge that mathematics teachers need and then discuss critical elements in designing professional development from a situative perspective.

PSC Goals: Enhancing Teachers' Professional Knowledge

Researchers and policymakers have come to agree that objectives for teacher learning should include becoming more proficient in the content they teach, gaining a better understanding of student thinking and learning, and improving their skills in content-based instructional practices (Secretary's Summit on Mathematics, 2003). These learning objectives provide the foundation for the PSC model of mathematics professional development.

In his seminal work in this area, Shulman (1986) identified subject-matter content knowledge and pedagogical content knowledge as two central domains of teachers' knowledge. Both domains are unique to the profession of teaching and can be enhanced over time as teachers gain expertise in their fields and participate in programs designed to foster such knowledge development (Wilson, Shulman, & Richert, 1987). Ball and her colleagues have extended Shulman's work in the field of mathematics education. Specifically, they have identified and elucidated "knowledge of mathematics for teaching" — the mathematical knowledge that teachers must have in order to do the mathematical work of teaching effectively (e.g., Ball & Bass, 2000; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2005; Hill & Ball, 2004). This conception of knowledge of mathematics for teaching is multifaceted and incorporates both content and pedagogical content knowledge.

Mathematics content knowledge. Ball and Bass (2000) describe the mathematics content knowledge needed for teaching as including "common" and "specialized" knowledge of mathematics. Common content knowledge can be defined as a basic understanding of mathematical skills, procedures, and concepts acquired by any well-educated adult. Specialized knowledge involves a deeper, more nuanced understanding of mathematical skills, procedures, and concepts. Specialized knowledge enables teachers to evaluate the multiple, and novel, mathematical representations and solution strategies that students bring to the classroom; to analyze (rather than just recognize) errors; to give mathematical explanations; to use developmentally appropriate mathematical representations; and to be explicit about their mathematical language and practices (Ball & Bass, 2003). It is what Ma (1999) characterizes as "profound understanding of fundamental mathematics" (p. 120).

Pedagogical content knowledge. Mathematics teachers need a sophisticated understanding of instructional practices and student thinking related to specific mathematical content. Ball and her colleagues consider these two types of understanding as distinct components of pedagogical content knowledge: knowledge of content and teaching, and knowledge of content and students (Ball, Thames, & Phelps, 2005). Knowledge of content and teaching includes, for example, the ability to recognize instructional affordances and constraints of different representations, and to sequence content to facilitate student learning. Teachers draw upon this knowledge when they plan for the use of pedagogical strategies and instructional materials in a lesson, when they modify a task or introduce a new representation during instruction, and when they consider how to improve their instructional practices the next time they implement a lesson with related mathematical content. Knowledge of content and students includes the ability to predict how students will approach specific mathematical tasks, and to anticipate student errors. Teachers draw upon this knowledge when they create lesson plans that take into account the thinking that a task is likely to evoke in their students, when they interpret incomplete student ideas during a lesson, and when they consider how to respond to the various correct or incorrect pathways that students explore.

Although these domains of knowledge of mathematics for teaching can be separated for purposes of analysis, they are inextricably intertwined in teachers' instructional practices. Teachers routinely make decisions that draw upon all aspects of their knowledge as they engage in the numerous and complex activities of classroom instruction — activities such as selecting, modifying, and using mathematical tasks; selecting mathematical representations that are appropriate for a specific learning goal and group of students; understanding and building upon student conceptions; and establishing and maintaining a discourse community that enhances students' mathematical understanding and their capacity to reason mathematically.

Whereas knowledge of mathematics for teaching includes all strands of school mathematics, our research and professional development as part of the STAAR project focused specifically on algebra; hence we use the term "knowledge of algebra for teaching" (KAT)[2]. Drawing upon the framework developed by Ball and colleagues, we conceptualize enhancing knowledge of algebra for teaching as enhancing both specialized content knowledge related to *algebraic reasoning* and pedagogical content knowledge related to *algebra instruction.*

Designing Professional Development from a Situative Perspective: Community and Artifacts as Tools for Teacher Learning

Situative perspectives on cognition and learning provide the conceptual framework that guided the design of the PSC. In the field of professional development, a situative perspective supports the value of creating opportunities for teachers to work together on improving their practice, and of locating these learning opportunities in the everyday practice of teaching (Ball & Cohen, 1999; Putnam & Borko, 1997; Wilson & Berne, 1999).

Professional learning communities. Situative theorists draw our attention to the social nature of learning and the central role that communities of practice can play in enhancing teachers' professional knowledge and improving their practice (Greeno, 2003; Lave & Wenger, 1991; Little, 2002; Putnam & Borko, 2000). To create an environment in which teachers collectively explore ways of improving their teaching and support one another as they work to transform their practice, successful professional development programs must establish trust, develop communication norms that enable challenging yet supportive discussions about teaching and learning, and maintain a balance between respecting individual community members and critically analyzing issues in their teaching (Frykholm, 1998; Seago, 2004). Research also indicates that the development of teacher communities is difficult and time-consuming work. Although conversations in professional development settings are easily fostered, discussions that support critical examination of teaching are relatively rare (Grossman, Wineburg, & Woolworth, 2001; Stein, Smith, & Silver, 1999).

Artifacts of practice. Another central tenet of situative perspectives is that the contexts and activities in which people learn become a fundamental part of what they learn (Greeno, Collins, & Resnick, 1996). This tenet suggests that teachers' own classrooms are powerful contexts for their learning (Ball & Cohen, 1999; Putnam & Borko, 2000). It does not imply, however, that professional development activities should occur only in K-12 classrooms. An alternative is to use artifacts of classroom practice—such as instructional plans and assignments, videotapes of lessons, and student work produced during a lesson—to bring teachers' classrooms into the professional development setting (Kazemi & Franke, 2004; Little, Gearhart, Curry, & Kafka, 2003; Nikula, Goldsmith, Blasi, & Seago, 2006; Sherin & Han, 2004). Such records of practice make the work of teaching a central focus of professional learning experiences and anchor conversations in specific classroom events.

Video records of classroom practice are becoming increasingly popular as a tool for teacher professional development. Short video clips can be selected to address particular professional development goals. They can be viewed repeatedly and from different perspectives, enabling teachers to closely examine one another's instructional strategies and student learning, and to discuss ideas for improvement. Although any video of classroom instruction can situate professional development in a setting that is likely to prove meaningful for teachers, there are conceptual and empirical arguments for using video from participants' own classrooms. Video from teachers' own classrooms situates their exploration of teaching and learning in a more familiar, and potentially more motivating, environment than does video from unknown teachers' classrooms (LeFevre, 2004). In one comparative study, teachers who watched video from their own classroom, in a computer-based professional development environment, found the experience to be more stimulating than did teachers who watched video from someone else's classroom, and they believed that the professional development program had greater potential for promoting instructional change (Seidel et al., 2005). The "video club" mathematics professional development program by Sherin and colleagues (Sherin, in press; Sherin & Han, 2004; Sherin & van Es, 2002) and the Video Case Studies in Scientific Sense Making Project by Rosebery and colleagues

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[2] The term KAT is also used by the *Knowing Mathematics for Teaching Algebra* project at Michigan State University (Ferrini-Mundy et al., 2005). These two projects are unrelated, although our work draws upon their conceptualization of knowledge of algebra for teaching.

(Rosebery & Puttick, 1998; Rosebery & Warren, 1998) informed our thinking about how to create an effective professional development program that incorporates video from participating teachers' own classrooms.

Establishing community around video. . Establishing and maintaining a strong community is particularly important when teachers are asked not only to discuss teaching and learning but also to share video clips from their own classrooms with colleagues. Because classroom video clearly exposes actual teaching practices, sharing video is likely to seem more threatening to teachers than sharing other artifacts such as student work and lesson plans. To be willing to take such a risk, teachers must feel confident that showing their videos will provide valuable learning opportunities for themselves and their colleagues, and that the atmosphere in the professional development setting will be one of productive discourse.

In an appropriate professional development setting, analyzing video from teachers' own classrooms can help to foster a tightly knit and supportive learning community. As teachers share video records of their teaching with colleagues, they have the opportunity to create an atmosphere of openness and bonding that is rare in professional learning environments (Sherin, 2004). Creating and maintaining a productive learning community around video is an integral component of our professional development model (Borko, Jacobs, Eiteljorg, & Pittman, in press).[3]

THE PROBLEM-SOLVING CYCLE

In this section, we describe the three workshops that make up one iteration of the PSC, discuss decisions central to planning each workshop, and identify some of the variations enacted by the STAAR team. In another paper, we provide vignette descriptions of each workshop from one iteration of the PSC, illustrating the opportunities teachers had for learning about mathematics content, pedagogy and student thinking (Koellner et al., in press). In addition, our website (http://www.colorado.edu/education/staar/) includes a Facilitator's Guide to Planning and Conducting the Problem-Solving Cycle. The guide is intended to help professional development facilitators learn about the Problem-Solving Cycle and prepare to implement it.

WORKSHOP 1: Doing for Planning

The major objective of Workshop 1 is to support the development of teachers' mathematics content knowledge. Most of the workshop time is devoted to teachers collaboratively working on the selected mathematical task and debriefing their solution strategies. Additionally, teachers spend a significant portion of Workshop 1 developing unique lesson plans that will meet the needs of their students. Specifically, they identify learning goals, predict student solution strategies, and structure their lessons with specific pedagogical moves. Teachers then implement their lessons prior to Workshop 2. Thus, another aim of the workshop is to enhance teachers' *pedagogical content knowledge* through discussions about designing a lesson plan and considering different ways of teaching the selected task. We call the framework for this workshop "Doing for Planning" to highlight the dual focus on teachers' problem solving and instructional planning.

Selecting the task. As described above, the PSC is built around a rich mathematical task. Teachers work through the task, design a lesson incorporating the task, teach that lesson to their students, and discuss their classroom experiences in two subsequent workshops. For the PSC to be successful, facilitators must select a task that can foster a productive learning environment for the teachers over the course of three workshops. In our development and implementation of the PSC model, we have found that appropriate tasks meet the following criteria: (1) address multiple mathematical concepts and skills, (2) are accessible to learners with different levels of mathematical knowledge, (3) have multiple entry and exit points, (4) have an imaginable context, (5) provide a foundation for productive mathematical communication, and (6) are both challenging for teachers and appropriate for students.

Given our focus on algebraic reasoning, for each iteration of the PSC conducted by the STAAR team, the facilitators sought problems that contained mathematical ideas central to the middle school algebra curriculum. Facilitators selected problems that focused specifically on the algebraic concepts of patterns and functions; enabled teachers and students to utilize different representations of functions such as graphs, tables, and equations; and had connections

[3] A discussion of the STAAR professional development program's approach to building professional community is beyond the scope of this article. For more information about the program, including our approach to building community, please see Authors 2005; and Authors, in press a.

to other areas of the mathematics curriculum such as number and operations, and geometry.

Conducting the workshop. The "Doing for Planning" framework guides the structure of Workshop 1. Teachers first read the selected problem and share ideas about the mathematical concepts and skills that are likely to be embedded in the solution strategies. They then work on the problem in small groups. During this time, the facilitator encourages the teachers to think about how they would create a lesson for their students incorporating the problem. At various points in the workshop, the teachers come together as a whole group to share their solution strategies and their ideas for using the problem in their teaching. As teachers create lesson plans tailored to their own students, they talk with colleagues and the facilitator about such issues as their mathematical goals for students, prior knowledge students will need for the lesson, and how they will adapt tasks to make them more accessible for their students. By the end of the workshop, teachers have explored the mathematical opportunities presented by the task, considered how their students might attempt to solve it, and developed a lesson plan for using the task in their classrooms..

Implementing and Videotaping the Lesson

Between Workshops 1 and 2, each participant teaches the problem in one of his or her mathematics classes, and the lesson is videotaped. In the STAAR program, we used two cameras to film each lesson. One camera followed the teacher throughout the lesson, and a second camera captured one group of students as they worked during small group activities. One of the most important components in Workshops 2 and 3 is the analysis of teachers' pedagogical moves and students' mathematical reasoning using video clips of the PSC lessons. Therefore, after the videotaping occurs, the facilitator selects short clips to serve as anchors for discussions about teaching and learning during Workshops 2 and 3.

WORKSHOP 2: Considering the Teacher's Role

The central purpose of the second workshop is to foster teachers' pedagogical content knowledge by guiding them to think deeply about the role they played in teaching the selected problem to their students. The majority of time in Workshop 2 is spent watching and discussing short video clips from one or more of the teachers' lessons, and exploring aspects of the teacher's role such as how they introduced the problem or orchestrated the classroom discourse. The workshop provides teachers the opportunity to critically reflect on their own instructional practices, along with those of their colleagues, as they analyze video clips and participate in guided discussions. The rich task and accompanying video situate the workshop in particular mathematical content and classroom practices, and this interaction between content and pedagogy is highlighted throughout the workshop

Planning the workshop. In planning for Workshop 2, the facilitator identifies one or more aspects of the teacher's role to explore. This decision depends on the particular needs and interests of the group of teachers as well as overall goals of the professional development program. Another key set of decisions for the facilitator involves selecting video clips to show and developing guiding questions for discussions during the workshop. We have found that video clips that work well in the PSC model have the following characteristics: (1) are relevant to the teachers, (2) are valuable, challenging, and accessible to the teachers, (3) cover a relatively short time period, and (4) provide an anchor for considering new instructional strategies. In addition, we have learned that it is important to prepare questions to help frame teachers' viewing of and conversations about each video clip.

During our three iterations of the PSC, the STAAR facilitators focused on topics related to the teacher's role such as introducing the task; posing questions to elicit, challenge, and extend students' thinking; deciding when to provide explanations, ask leading questions, and let students follow their own line of reasoning; and wrapping up the lesson.

Conducting the workshop. Workshop 2 typically begins with teachers reflecting on and sharing their experiences teaching the problem. Subsequent activities are designed around the selected pedagogical topic and associated video clips. Teachers view the clips in both small group and whole group contexts, and the facilitator guides conversations about the instructional episodes they capture. Often, a video clip is viewed multiple times, as the conversation suggests another perspective to take or another interpretation to explore. Teachers are also given time to reflect critically and to consider ways of improving their instruction that they can take back to their classrooms.

WORKSHOP 3: Considering Student Thinking

The central objectives of Workshop 3 are to deepen teachers' understanding of students' thinking about the mathematics in the selected PSC task, and to extend their ideas about how to foster and support students' mathematical reasoning. To situate teachers' explorations in their classroom practice this workshop relies heavily on clips from the videotaped lessons as well as additional artifacts that represent student thinking, such as students' written work and reflections. Throughout the workshop, teachers have opportunities to gain further insight into the complexities of both the mathematical concepts entailed in the problem and students' learning of those concepts.

Planning the workshop. A major task in planning Workshop 3 is selecting artifacts of practice that will provide opportunities for teachers to explore the various forms of mathematical reasoning their students applied to the problem and the different solution strategies they used. To select video clips, the facilitator considers the same characteristics as in planning for Workshop 2; however, rather than choosing clips to provide an anchor for examining instructional strategies, the facilitator selects clips to provide an anchor for considering student thinking. In a similar manner, facilitators select "rich" examples of student work such as individual student work on the task, posters created by groups of students, and written reflections. As in Workshop 2, the facilitator prepares guiding questions to help frame teachers' conversations about each video clip and example of student work, encouraging them to focus on the mathematical concepts and reasoning evident (or lacking) in the artifact.

The STAAR facilitators often chose video clips and student work that featured novel ways of solving the mathematical problem—in particular, solution strategies that none of the teachers noted during Workshop 1. We also addressed topics such as how students explained their solution strategies, and misconceptions or naïve conceptions.

Conducting the workshop. . In Workshop 3, teachers spend the majority of the time watching and discussing video clips and students' written work. Close analysis of the mathematical content in the clips and other artifacts often leads the teachers to rework the problem, and to engage in mathematically sophisticated conversations. For example, they may be prompted to discuss the affordances and constraints of various solution methods, the progression from naïve to more formal understandings of the content, and mathematical ideas embedded in the problem that they had not previously considered. Workshop 3 also includes time for teachers to reflect on what they have learned, in this workshop and over the course of one iteration of the PSC. As they reflect, individually (in writing) and collaboratively (in small or whole group discussions), the teachers not only consider how they might improve their instructional practices based on knowledge gained thus far but also provide valuable input that the facilitator can use to shape successive iterations of the PSC.

RESEARCH METHODS

The STAAR professional development program began in 2003 and continued through spring 2005. During that time, we worked with a group of middle school mathematics teachers to develop and refine the PSC model. In fall 2003, we conducted three professional development workshops that focused on pedagogical practices associated with algebra. A central goal of these workshops was to develop norms for viewing and analyzing classroom video before conducting the first iteration of the PSC. We conducted the first PSC in spring 2004 and two more iterations during the 2004–2005 academic year. The three iterations used different mathematics problems and focused on different aspects of the teacher's role and students' mathematical reasoning. During the three iterations of the PSC, we utilized a design experiment approach (Cobb et al., 2003; Design-Based Research Collective, 2003) to study and refine the model

Participants

Eight teachers participated in the STAAR professional development workshops during the 2003–2004 academic year. All eight were middle school mathematics teachers, with classroom experience ranging from 1 to 27 years. They represented six different schools in three school districts within the state. In 2004–2005, seven teachers continued working with us and three additional teachers joined the project, as we further refined the PSC. Each new teacher was a colleague of one of the current participants.

Data Collection and Analysis

Throughout the professional development program we collected and analyzed a large amount of data on processes involved in developing and enacting the PSC model (see also Borko et al., in press and Koellner et al., in press). We also collected data on the teachers' experiences and learning outcomes over the course of the two years that they participated in the STAAR program. At the end of the second year we conducted both a written survey and individual face-to-face interviews asking the teachers to consider the impact of the professional development program on their learning of algebra, beliefs about learning and teaching algebra, and instructional practices. In addition, we conducted a follow-up interview with each teacher during the school year following the conclusion of the professional development workshops, in order to assess their perception of the continuing impact of the professional development program.

To examine teachers' perceptions of the impact of the program, two coders analyzed three sets of self-report data: the written surveys completed during the final PD workshop, the post-program interviews conducted shortly after the final PD workshop, and the follow-up interviews conducted the next academic year. All of the interviews were transcribed. The coders independently marked all instances where the teachers wrote about or discussed the following categories:

- Impact on content knowledge,
- Impact on pedagogical content knowledge related to the teacher's role,
- Impact on pedagogical content knowledge related to student thinking, and
- Impact of watching video (including video of themselves and of their colleagues).

The coders then met to discuss and reconcile their coding decisions. In our analyses we report on the number of teachers who brought up these categories in at least one of the three data sources.

IMPACT OF THE STAAR PROFESSIONAL DEVELOPMENT PROGRAM

In this section we present initial results regarding the impact of the STAAR professional development program on the participating teachers' professional knowledge from their perspectives. In particular, we illustrate the perceived impact of the program on teachers' mathematics content knowledge and pedagogical content knowledge — specifically the teacher's role in promoting discourse and student thinking. We also explore the teachers' perspectives regarding a central component of the PSC model: watching video of themselves and their colleagues.

Impact on Content Knowledge

Analyses of the three self-report data sources suggest that the teachers believed their content knowledge was fostered through participation in the professional development program. Specifically, we examined three categories of coded data related to content knowledge: a) learning mathematics content (generally), b) learning by working on the mathematics tasks that were part of the PD, and c) learning from using multiple approaches to solve the mathematics tasks. Looking across the three data sources for the eleven teachers[4] who participated in the program, six teachers mentioned learning mathematics content, all eleven mentioned learning by working on the mathematics tasks, and ten mentioned learning from using multiple approaches (see Table 1).

Impact on Pedagogical Content Knowledge

We considered the impact of the professional development program on two aspects of teachers' pedagogical content knowledge that are emphasized heavily in the PSC model: the teacher's role and student thinking.

Knowledge about the teacher's role in promoting discourse.

Based on the participants' stated interests and instructional goals, the STAAR project focused on the teacher's role in improving classroom discourse. Therefore, in our analyses, we coded the three sources of self-report data for teachers' perceptions of the impact of the PD on their role in promoting discourse. All eleven teachers reported that the program helped to increase their knowledge about promoting classroom discourse, including learning about the importance of meaningful discussions and techniques for fostering discussions (see Table 2). Most of the teachers talked about the program as having an impact on specific aspects of their knowledge about classroom discourse. For example, ten teachers noted that they learned something about conducting groupwork, such as how important it is to provide time for groupwork or how to group their students more effectively. Nine teachers said that they learned how to foster better conversations in their mathematics classrooms, either within small groups or during whole class discussions. Eight teachers mentioned that they learned something about asking questions, including what types of questions are most effective and strategies for asking questions to elicit student thinking.

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[4] The data discussed in this section are from eleven teachers: seven participated in both years of the professional development program and 4 participated in one year.

Knowledge about student thinking.

Teachers' comments on all three self-report data sources suggest that participation in the PSC strongly impacted their knowledge related to student thinking. All eleven teachers commented that they gained a general awareness of students' mathematical thinking, including learning about how to listen to and promote their students' thinking (see Table 3). In addition, all of the teachers said that they learned about the importance of giving students more authority, for example by making their classrooms more student-centered or by decreasing their own role as the mathematical authority. Eight teachers said they became more knowledgeable about how to use or build on their students' mathematical thinking. Seven teachers reported learning about how to use mathematical tasks to promote student thinking, such as using rich problems that emphasize exploring processes rather than generating answers, or using fewer problems and exploring them for a longer period of time.

Impact of Watching Video

Because watching video is such a prominent feature of the PSC — and new to most teachers — we wanted to examine participants' perspectives on the value of this component of the professional development program. We coded and analyzed the teachers' self-report data to explore how they felt about watching video from their own lessons and from their colleagues' lessons.

Watching video of themselves. Ten teachers told us that being videotaped, although sometimes nerve-racking, was one of the most valuable aspects of the professional development. Many of these teachers pointed out that watching their own lessons on videotape enabled them to see what they were doing well and to identify areas for improvement. A number of teachers commented that by watching video they gained insight into what their students were thinking and what assistance they needed..

> *I was filled with anxiety when I thought someone was going to come in and videotape everything I was doing during classes with kids. But it turned out to be a powerful learning experience for me.* (Pam, post interview)

> *Watching the video clips was great to see me in action and actually get to see what the students see. It allowed me to see the parts of my lessons that need improvement and what is good.* (Laura, written reflection)

> *I think the most helpful [thing] was the videotaping, to watch myself on videotape, sometimes painfully so. Wanting to say, "Shut up, shut up. Why do you keep going on with that?" But it's so helpful to see how you come across to kids and how they are or are not responding … and to think about what I might have changed in that lesson … or how I could have connected with kids better.* (Celia, final interview)

Watching video of other teachers. . Eight teachers mentioned that they learned something by watching videos of their colleagues, such as new pedagogical strategies or how other students solve mathematical problems. Several teachers mentioned that it was informative as well as reassuring to watch their colleagues struggle with familiar issues.

> *We never get to see our colleagues doing what we're doing. We just assume they're doing the same things that we are, and that's not necessarily so. It's a great window into how other kids look and it's comforting when you see things that are the same.* (Penny, final interview)

> *When I watched other teachers' videos, it wasn't critiquing, it was seeing what they do in their classroom and realizing [that] a lot of what's going on in their classroom is what's happening in mine. Or … this person really does a great job at opening a lesson. Maybe I could try something they're doing.* (Linda, final interview)

CONCLUSION

As this sample of findings from research on the STAAR professional development program illustrates, the Problem-Solving Cycle appears to be a promising model for enhancing teachers' content knowledge and pedagogical content knowledge. Our data suggest that as teachers engage in the PSC, they are prompted to think deeply about mathematics content and instruction as part of a collaborative and supportive learning community. In particular, teachers who participated in the STAAR program report a strong impact on specific areas of their professional knowledge that were targeted during the three iterations of the PSC: mathematics content (including the importance of working on tasks and generating multiple solution strategies), methods for improving classroom discourse (including how to conduct groupwork, foster conversations about mathematics and mathematical thinking, and ask effective questions), and ways of fostering and exploring student thinking (including giving students authority, building on students' thinking, and using tasks that promote student thinking).

The PSC model provides a structure for the participating teachers to work together as professionals, to establish trust and develop communication skills that enable constructive yet respectful discussions about teaching and learning, and to share and expand their knowledge base. Drawing on a situative framework, the model emphasizes the use of classroom artifacts within a supportive professional community. Any professional development effort that foregrounds the analysis of video from teachers own classrooms is entering into relatively uncharted, and murky, territory. However, our experience suggests that when the necessary structure is in place, the impact on teachers can be extremely powerful and fundamentally positive.

We are particularly encouraged by the fact that teachers at four of the six schools represented in the STAAR project are spearheading new professional development efforts within their schools that contain some or all of the PSC elements. Teachers at several schools have decided to observe and videotape one another and then meet to discuss these videotapes. At one school, the mathematics instructors plan to all work on and then teach a selected problem, and get together to share their experiences. When asked about their reasons for initiating these professional development activities, the teachers explained that they felt empowered by their experiences in the STAAR program and wanted to share what they had learned with their colleagues. The following remarks, from three of the teachers' final written reflections, are illustrative of these ideas:

> *I proposed this sort of "community" to my principal and next year we will meet once a week as grade-level math departments. The problem is that teachers have a mentality of "shut the door and let me teach." I hope my school's math people can get the same sense of community as we have here.* (Peter, written reflection)

> *I have learned to become a leader in my professional community. I have been able to share my classroom with other teachers so they can take ideas about teaching and learning from me.* (Nancy, written reflection)

> *I want to get my entire department involved with this process. As we put students in groups to work together, we as teachers need to do the same. We need to be doing math together.* (Laura, written reflection)

Although our implementation of the PSC has been restricted to middle school teachers and focused on algebra content, the model is intentionally designed to be flexibly implemented and responsive to the needs of facilitators, teachers, and school district personnel. We anticipate that it can be adapted for use with teachers at elementary and high school levels and with different strands of the school mathematics curriculum. While our research and development work on the PSC model will continue, we encourage others in the mathematics education community to adapt, extend, and refine this approach and further explore its effectiveness.

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TABLE 1. *Perceived impact of the professional development program on teachers' content knowledge*

	Number of Teachers	Representative Quotes
Learning mathematics content (generally)	6	One of the things I was really weak at was trying to develop equations from patterns. I just could not do that for the life of me before [the STAAR program]…. I actually forced myself to use those strategies… and it's really beginning to open my eyes. *(Nancy, final interview)* I used to think in algorithm mode. Now I try to see or picture patterns. *(Deborah, written reflection)*
Learning from tasks	11	[I learned] just how much insight you can get from working problems with other adults… When you do it on your own, you've got a much narrower view on it to start with. Whereas if you solve it with other adults before teaching, it broadens your view. And then the kids broaden it even more. *(Kristen, final interview)* Before the STAAR workshops, I have to admit honestly that I did not try every single new rich problem, or non-rich problem, myself all the time. I'd look at the parameters of the problem, but not necessarily sit down and work them. You know what the STAAR project taught me? Feel their pain. Look at the problem, and work it either yourself or with someone else. *(Pam, post interview)*
Learning from multiple approaches	10	I have learned that there's more ways than I could have imagined to solve problems. Without this program I would not have realized all the ways to solve a problem and the importance of looking at student work and thinking. *(Linda, written reflection)* This group has enhanced my algebraic knowledge by listening to others' ideas to the same problem. Learning that multiple solutions do exist and [that it's important] to study them purposefully with kids. *(Penny, written reflection)*

TABLE 2. *Perceived impact of the professional development program on teachers' knowledge about their role in promoting discourse*

	Number of Teachers	Representative Quotes
Teacher's role in discourse (general)	11	I no longer think that math class is about me. It's about them and their learning. And it's about my facilitating… I can say, 'OK. Let's look at this and talk about it.' *(Celia, final interview)* I think learning to struggle is as important as anything else in math. [STAAR] helped me to know that because you put me through it! Now when kids say to me in class, 'Well I can't do it. Give me a hint,' I say, 'Maybe you better go talk to your group.' I step back and I step back for a good long time until we bring the large group back together again. *(Pam, final interview)*
Conducting groupwork	10	Working with other teachers on the math problems was really beneficial. And that led me to understand why it's so important for students to work in groups in the classroom. *(Linda, final interview)* I used to just kind of let the kids pair up and I didn't have much thinking behind it. Now I structure it and have a purpose between who's with whom. *(Peter, post interview)*
Fostering conversations	9	I realized the importance of talking about our thinking, and giving kids the opportunity to share their ideas. (Ken, written reflection) I now try to say to them, 'Please share that with the rest of your group…. Explain that to everyone.'…. I want them to pursue that and ask those questions of each other. *(Celia, final interview)*
Asking questions	8	I feel like I went from a lot of lecture … a very broad kind of questioning [style] to asking deeper level — or higher level — thinking questions, provoking more of their thinking, rather than just "Is this right?" kinds of questions. *(Ken, final interview)* I try to get them to explain more about what they did. And so my questions ask for more than the answer. 'How did you get there?' 'Why did you do it that way?' 'Does anyone else want to tell us how they did it?' Those kinds of things. I think it's all about digging in deeper. *(Kimber, post interview)*

TABLE 3. *Perceived impact of the professional development program on teachers' knowledge about student thinking*

	Number of Teachers	Representative Quotes
Awareness of student thinking (general)	11	I thought so much about looking at kids' work and trying to figure out what they were thinking with the STAAR program. *(Kimber, post interview)* Watching other teachers allow their students to think and discover and digest a problem makes me realize that is a change I must make. *(Kristen, written reflection)*
Giving students authority	11	I learned how to not just tell students how to do things, but have them participate… and share their information. Instead of me just standing up there and blabbing the hour and a half. *(Linda, final interview)* I don't want to keep pushing them to get my answer and to follow my path. I want them to find their own path. *(Kristen, final interview)*
Building on student thinking	8	Previously, I wouldn't allow my students to continue their thought process. I would stop them and have them go my way. Versus now, when I'm not quite understanding what they're doing, I will continue to ask questions. *(Laura, post interview)* Before STAAR I would have said immediately, 'Oh yeah, that's right. Move on.' Now we explore it deeper than that. And they know, too, that I'm going to say to them, 'How can you prove it?' *(Celia, post interview)*
Using tasks to promote student thinking	7	Student thinking takes time. [This knowledge has] helped me determine what part of the curriculum is more important, so I can do away with 'less important' problems. *(Nancy, written reflection)* STAAR showed me that there are problems out there that have so many things to offer kids. So many things that they can talk about and experience and try to strategize. *(Pam, final interview)*

An Agent of Change:
NSF Sponsored Mathematics Curriculum Development

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ABSTRACT:
This article identifies factors that make it difficult for publishers of commercial textbooks to make significant changes consistent with curricular visions put forth by the National Council of Teachers of Mathematics (NCTM). Central among these factors is the lack of consensus of state standards on what and when certain topics in mathematics should be addressed. The variability of grade placement of key mathematics learning goals across different state standards results in excessive repetition and superficial treatment of topics in school mathematics textbooks.

In response to the NCTM release of their Curriculum and Evaluation Standards for School Mathematics *(1989) the National Science Foundation funded 13 different projects to construct mathematics curriculum materials consistent with the vision put forth in the document. During the last 15 years innovative curriculum materials were developed, piloted, revised, and are now used in many schools throughout the United States. Some have become more popular than others, but collectively they comprise between 10-20 percent of the mathematics textbooks being used today in K-12 classrooms.*

This article highlights the impact of this massive effort to bring about change in mathematics teaching and learning in K-12 schools. In addition to the impact on students' mathematics learning, the new mathematics curriculum materials have also influenced teacher practice as well as the professional growth and development of classroom teachers. The availability of comprehensive innovative curriculum materials consistent with the vision of NCTM has stimulated an enormous amount of research in schools, and influenced textbooks being developed by commercial publishers. It has also become a political issue that has stimulated discussions about mathematics curriculum involving a wide range of constituents at the state and national levels.

The National Science Foundation's effort to stimulate the development of mathematics curriculum materials (textbooks) based on a new model — one that relies on a cycle of curriculum design, implementation, and refinement based on field trials — has stimulated discussion, collaboration and action through the world of textbook publishing. The effort produced an array of different K-12 mathematics textbook series, K-12 (see Table 1). Some might describe the impact of NSF support for curriculum development as a 'ripple' within a large ocean of the textbook publishing world. Others, a 'wave' that significantly impacted a small group of schools, teachers and students. Still others might view the result as a massive wave — changing the very landscape of textbook publishing and implementation. The ultimate impact is likely too early to know.

CHALLENGES FOR SCHOOL TEXTBOOK PUBLISHERS

It seems reasonable that textbooks sold and used by millions of K-12 students and their teachers should be carefully researched by the authors and publishers prior to their distribution to insure that they are effective resources in helping students learn mathematics. However, historically mathematics textbooks have not been researched and piloted before being sold commercially [Tyson-Bernstein, 1988]. The challenges of developing research based mathematics textbooks for K-12 schools were discussed by Willoughby:

> "A carefully conceived, well-written textbook takes several years to develop. A well-written, adequately tested textbook series takes much longer, since the books in the series ought to be field tested longitudinally (one grade at a time), if anything really new is being done. The only way publishers can satisfy the insatiable demand by adoption committees for the latest thing is to fake it." [Willoughby, 2002, p. 141]

Willoughby's description applies to the majority of the K-12 mathematics textbooks that are used in schools today. Whitman echoes the same theme as he describes how some textbooks are rapidly assembled within development houses [Whitman, 2004]. While recognizable names of authors are visible on the cover of textbooks, these authors may have played very minor roles in writing the materials. Whitman points out that the constant demand for textbooks with new copyright dates precludes publishers from field-testing their K-12 products over several years to study their impact on student learning.

The production of textbooks is a very big business in the United States and a variety of factors work against careful research and development efforts by commercial publishers. For example:

1. No common set of mathematics curriculum standards exists in the USA. Although the National Council of Teachers of Mathematics, and other groups such as Achieve and the College Board, have proposed standards there is no national consensus on what students should learn or when they should learn it. Instead, most states develop and require school districts to follow state curriculum frameworks which specify learning goals by grade or by course.
2. As a result of the lack of agreed upon standards, there is wide variation in the placement of topics in current mathematics textbooks. For example, one state may expect students to become fluent with multiplication facts in grade 2, whereas another state may expect fluency in grade 4 (Reys, 2006). Thus, textbooks sold in each of these states must include the same topic in multiple grades.
3. Short timelines imposed by state textbook adoption committees often preclude thoughtful and research-based development of textbooks. It takes years to develop a new textbook series, and more years to field-test its effectiveness. Yet states often issue their standards or framework within one or two years of their adoption deadline. This tight timeline makes any longitudinal research study of the impact of textbooks on student learning impossible to implement.
4. Many teachers are resistant to significant changes in curriculum and textbook format. There is comfort and security in using the same textbook for several years because teachers are familiar with the order of content and often have well established lessons. Textbook sales representatives capitalize on this comfort of using the same textbook by 'rolling over' current users of one edition to the next edition of their textbooks. Teachers are already familiar with the material, and it requires little or very limited new learning to implement the new edition.

Despite these factors, new K-12 mathematics textbooks are published regularly. They are new in copyright and tend to incorporate features of mathematics curricula that have the largest market share and thus significant change is rare. As a result few of these textbooks are new in the sense of having different content, format or style. Historically, commercial publishers of textbooks have been unwilling to commit significant resources to develop mathematics textbooks that differed significantly from those textbooks that were already successful in the market place [Reys & Reys, 2006]. The most obvious changing feature of mathematics textbooks has been their growth in size. This is reflected in lengthy textbooks, often exceeding 700 pages. As noted earlier, variability in the standards or learning goals across states is a major contributor of the growing size of textbooks. Consequently, publishers cover the standards of multiple states in the same textbook, and a significant amount of content is duplicated from grade to grade. Often the duplicated content receives shallow or superficial treatment in multiple grades, resulting in the characterization of the mathematics curriculum in the United States as being "a mile wide and an inch deep" (Schmidt, McKnight, and Raizen, 1997).

A LARGE-SCALE EFFORT TO CHANGE MATHEMATICS CURRICULUM MATERIALS

Change is slow, more similar to ocean tides gradually changing the landscape, than significant changes resulting from a tidal wave. Ripples result when still water is disturbed. It is safe to say that the landscape of K-12 mathematics curriculum was 'still water' in the 1980s,

TABLE 1. *Comprehensive mathematics curriculum development projects funded by NSF*

Project Name	Grades	Curriculum Development Sponsor	Initial Commercial Publisher
Investigations in Number, Data and Space	K-5	TERC	Scott Foresman
Math Trailblazers	K-5	University of Illinois-Chicago	Kendall/Hunt
Everyday Mathematics	K-5	University of Chicago	SRA/McGraw Hill
MATH Thematics	6-8	University of Montana	McDougal Littell
MathScape: Seeing and Thinking Mathematically	6-8	Education Development Center	Glencoe/McGraw Hill
Mathematics in Context	5-8	University of Wisconsin	Holt, Rinehart, & Winston
Connected Mathematics Project	6-8	Michigan State University	Prentice Hall
Middle School Mathematics Through Applications (MMAP)	6-8	Institute for Research on Learning	Unpublished
Contemporary Mathematics in Context (Core-Plus)	9-12	Western Michigan University	Everyday Learning Corporation
Math Connections	9-12	MathConx	IT'S ABOUT TIME, Inc.
Systemic Initiative for Montana Mathematics and Science (SIMMS) Integrated Mathematics	9-12	Montana Council of Teachers of Mathematics	Kendall/Hunt
Interactive Mathematics Program (IMP)	9-12	Sonoma State University	Key Curriculum Press
Mathematics: Modeling Our World (MMOW/ARISE)	9-12	COMAP	W.H. Freeman & Co.

only occasionally disturbed by emerging technology. In 1989 the National Council of Teachers of Mathematics published the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). This document provided a new vision for K-12 mathematics and it resulted from years of work that was supported by more than 25 professional organizations, including organizations composed mostly of mathematicians, such as the American Mathematical Society and the Mathematical Association of America.

One of the challenges the NCTM *Standards* presented was the development of new curriculum materials for mathematics teaching that would better support student learning. Given the history of commercial publishers being uneasy about risking millions of dollars to develop a textbook series that is significantly different from the market leaders, it seemed unlikely that new mathematics textbooks reflecting the vision of the NCTM *Standards* would be forthcoming.

The National Science Foundation, concerned with mathematics performance reported by National Assessment of Educational Progress and the consistently low performance on international assessments since 1970, made a decision to support the development of research based K-12 mathematics textbooks. The NSF realized that the vision put forth by the NCTM *Standards* might take many different forms, and ultimately funded 13 different projects that spanned K-12 (shown in Table 1) (Reys, et al., 1999). These curricula were extensively field-tested in schools and then revised before becoming commercially available. The resulting mathematics curricula represent notable exceptions to traditional textbooks that typically lack a research and development phase prior to release (Trafton, et al., 1999).

The NSF-supported mathematics textbooks have been reviewed by committees of the US Department of Education and AAAS (Kulm, et al., 1997) and judged of exemplary quality compared to other commercially available textbooks. Studies have consistently reported positive growth in the mathematics learning, particularly related to reasoning and problem solving, as a result of use of the new curriculum materials (Senk & Thompson, 2003).

One testimony to the impact of NSF's effort is that tens of thousands of children are using these textbooks every day in schools throughout the United States. In some places NSF-supported mathematics textbooks are used by all schools in a district. In other places, teachers are using units or modules to supplement their current mathematics textbook. Estimates of the market share of NSF-supported textbooks range from 10-20 percent of students and teachers, indicating that the impact is more of a wave than a ripple (Education Market Research, 2006). Significant use of these textbooks is evidence that NSF's effort to stimulate new models of textbooks has been successful. However, the story does not stop there.

FAR REACHING IMPACT OF NSF-SUPPORTED MATHEMATICS CURRICULUM DEVELOPMENT

The mathematics curriculum materials produced with NSF support have provided a wide range of K-12 curricular options for students, teachers and schools. However, NSF's initiative has extended beyond school users, generating healthy discussions and some unanticipated by-products. While the impact may be short of a tsunami, it has generated significant waves in different directions including teacher development, teacher practice, research activity, and the textbook publishing industry. In addition, it has stimulated increased attention to K-12 school mathematics by the community of research mathematicians.

Impact on Professional Development Activitites

As teachers began using these K-12 NSF-supported curriculum materials it became clear that many were unable to implement the programs in the spirit that the authors intended. In some cases, teachers lacked the necessary content knowledge in mathematics to respond to questions their students asked. In other cases, teachers were uncomfortable with the active involvement of students in groups, and classroom management issues surfaced. As a result, professional development specifically organized to support teacher learning is essential (Ball, 1996). Developers of the curriculum materials and others have organized and provided professional development to strengthen teachers overall mathematical knowledge as well as their pedagogical expertise. In addition, many teacher education institutions focused attention on the K-12 curriculum materials to prepare new teachers (Papick, et al., 1999).

Impact on Teachers' Practice

A number of initiatives — local, statewide and national — have emerged to support the professional development of mathematics teachers using the NSF-supported curricula. As a result, many teachers have changed from teacher-centered to student-centered instruction. Thus students assume a greater responsibility for learning and helping their peers. There is growing research that shows teacher's knowledge of mathematics is also growing from their use of mathematics curricula. These teachers are learning mathematics as they teach (Remillard, 2005].

Impact on Students' Perception of Mathematics

Students have often viewed mathematics as a spectator sport. That is, mathematical procedures are demonstrated and then the procedure is practiced, and often these procedures are devoid of any meaningful context or focus on understanding. Consequently, memorization rather than sense making is associated with mathematics learning. As a result, many students and parents don't understand mathematics and what developed an unhealthy and distorted view it.

The NSF-supported materials embed mathematical concepts and skills in problem solving contexts. Although the learning activities are challenging, it is rare to hear students ask 'When are we going to use this?' as the context reflects challenging problems that are embedded within a realistic setting. As a result a higher percent of students engaged in these mathematics curricula at the secondary level are choosing to take more mathematics classes in high school (Harwell, et al., in press).

Impact on Commercially Developed Textbooks

Imitation is said to be the highest form of flattery. An examination of recently produced mathematics textbooks by commercial publishers shows that some problems and approaches used in the NSF-supported mathematics curricula are surfacing in commercially developed textbooks (Reys, et al., 2004). Adopting and adapting some of the ideas put forth in the NSF-supported mathematics curricula by commercial publishers is one of the indirect ways that the NSF-supported projects have impacted the larger spectrum of mathematics textbooks. Since commercially developed mathematics textbooks tend to be widely used in middle and secondary mathematics programs, their inclusion of more interesting, rich and challenging problems reflects a major impact from the NSF-supported mathematics curricula.

Impact on Research in the Field

NSF-supported mathematics curricula have been the focus of much research in the mathematics education community. Some research has been done as part of the research and development model that each of the curriculum projects followed. In addition, many research studies have investigated the impact of NSF-supported mathematics curricula on student learning as well as on teacher use. In fact, a review of articles reporting student learning outcomes related to mathematics curriculum in the *Journal for Research in Mathematics Education* from 1996-2006 reveals that over 80% (14 out of 16) involved NSF-supported mathematics curricula. This predominance of mathematics curriculum research involving NSF-supported mathematics textbooks is also reflected in doctoral dissertations. Research on mathematics curriculum has addressed many different issues including curriculum analysis, teachers' use of curriculum materials, and student and teacher learning associated with curriculum materials.

Increased Involvement of Research Mathematicians in K-12 Mathematics Programs

More mathematicians have become interested in K-12 mathematics programs. Some have expressed concern about changes occurring in the K-12 mathematics curriculum (Wu, 1997). Other mathematicians have taken opposing views in support of many of the changes (Kilpatrick, 1997; Cuoco, 2003; Ralston, 2004). These 'tugs of war' provide opportunities for healthy debate and constructive dialogue. However, reasoned debate has not always been the norm. Thus, in some circles, mathematicians and mathematics educators are viewed as holding opposite views and advancing different agendas. In fact, this is an overgeneralization as many mathematicians and mathematics educators share common goals and work together to develop and implement strategies to support the improvement of school mathematics programs.

Politics and Policy

The NSF-funded mathematics textbooks provide a clear alternative to traditional textbooks that are commercially developed. They also provide the basis for enacting a different vision for teaching and learning. Thus, the textbooks themselves are often the impetus for philosophical clashes between reform and anti-reform groups. For example, at the state level it was reported that "California's mathematics policy followed a persuasive (albeit deceptive) campaign alleging the failure of the current reform movement in mathematics education" and the NSF-supported mathematics curricula were at the epicenter of these discussions (Jacobs and Becker, 2000). The anti-reform movement was led by organized groups of politically savvy individuals who knew how to influence policy. The story of one school textbook adoption committee was recently chronicled and illustrates the range of issues and personal biases that surfaced, how opinions were persuaded, the value attached to research evidence and ultimately how decisions were made (Newman, 2004). The story is a reminder that "decisions about educational reform are driven far more by political considerations, such as the prevailing public mood, than they are by any systematic effort to improve instruction" or learning (Dow, 1991).

SUMMARY

A careful review of the impact of NSF mathematics curriculum development initiative over the last two decades must look beyond the number of textbooks sold and the number of students and teachers using the textbooks. The initiative has:

- influenced the mathematics content that students in the United States have an opportunity to learn;
- fostered the belief that mathematics learning should be meaningful and that learning mathematics should be a sense making experience;
- helped teachers increase their knowledge of mathematics;
- helped teachers establish more effective ways of helping their students learn mathematics;
- influenced commercially developed textbooks to incorporate mathematical problems, activities, ideas and developmental approaches based on an active learning model;
- encouraged mathematicians to become involved in reviewing and shaping mathematics textbooks; and
- stimulated an unprecedented wave of research activity focusing on the impact of mathematics curricula on teachers and students.

Focusing on an element of the educational system as basic as textbooks, used by virtually every teacher and student in the country on a daily basis, provides a powerful means of promoting change in practice. While time and continued monitoring of the field will tell the story of the true impact of this effort, there is clear evidence that NSF-supported curriculum innovation has generated more than a ripple or wave of change.

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State Mathematics Curriculum Standards and Reasoning

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Since the release of the National Council of Teachers of Mathematics document, *Curriculum and Evaluation Standards for School Mathematics,* in 1989, states began to develop their own standards to set expectations for their students. Recently, many states updated their standards to incorporate new demands and trends in mathematics education (see Reys, 2006). In an effort to assess reasoning expectations in the state mathematics curriculum standards (state standards hereafter), we reviewed 35 state standards from kindergarten to eighth grade (Authors, 2006). In doing so, we focused on the extent and nature of emphasis on reasoning in five content areas (i.e., number and operations, algebra, geometry, measurement, and data analysis and probability), how grade level expectations (GLEs) related to reasoning are organized in the state standards, and overall characteristics of emphasis on reasoning across state standards. In this paper, based on the results of the review we discuss expectations we can have from the state standards in terms of reasoning as well as issues to consider in order to better promote reasoning. Let us begin our discussion by elaborating what reasoning is and what we mean by reasoning in this paper.

Reasoning and Its Importance

There seems to be a wide agreement on the importance of reasoning in mathematics teaching and learning. Reasoning is a process standard emphasized throughout the NCTM documents (NCTM, 1989, 2000). Ball and Bass (2003) state that mathematical learning cannot be considered without reasoning. To reason mathematically is fundamental to learning mathematics with understanding. When reasoning is effectively promoted through justifying results, developing ideas, predicting results, or making sense of observed phenomena, students can develop a deeper understanding of mathematical ideas. In turn, this deeper understanding equips students to enhance their mathematical reasoning. This way of learning mathematics will result in better learning outcomes. In this sense, the NCTM argues, "Reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts" (2000, p. 56).

While people agree that reasoning is important in the teaching and learning of mathematics, as Duval (1998) argues, there seems to be a wide range of ideas on what reasoning means. Reasoning is a broad and general term. According to Duval, "[A]ny process which enables us to draw new information from given information is considered as reasoning" (p. 45). Because of this broadness of reasoning, researchers, curriculum developers and teachers interpret reasoning diversely. For example, *Principles and Standards for School Mathematics* (NCTM, 2000), which includes reasoning and proof as one of the five process standards, emphasizes the importance of: making and investigating mathematical conjectures; developing and evaluating mathematical arguments; and selecting and using various types of reasoning and methods of proof. The *Trends in Mathematics and Science Study* (TIMSS)'s assessment framework on reasoning includes the following elements: 1) hypothesize/conjecture/predict, 2) analyze, 3) evaluate, 4) generalize, 5) connect, 6) synthesize/integrate, 7) solve non-routine problems, and 8) justify/prove (Mullis, Martin, Smith et al., 2001). When analyzing mathematics curricula in terms of reasoning, Stylianides and Silver (2004) focus on the process of proving, that is:

TABLE 1: ***Number of state curriculum documents that include GLEs in content strands in each component of reasoning by grade***

	K	1	2	3	4	5	6	7	8	K-8
Prediction	8	17	24	24	24	26	22	25	27	35
Generalization	2	1	8	5	9	10	10	12	12	21
Verification	2	1	2	5	7	7	6	6	13	21
Justification	1	1	8	12	14	23	20	19	24	31
Conclusion/Inference	1	6	9	12	13	16	15	16	17	26
Making Conjecture	0	0	1	2	5	7	6	13	10	19
Testing	1	1	4	6	12	10	6	9	7	20
Making Argument	0	0	1	0	2	6	3	7	11	14
Evaluation	0	0	3	2	2	7	9	9	14	20

identifying a pattern, making a conjecture, providing a proof, and providing a non-proof argument. Ball and Bass (2003) view reasoning as a process of inquiry and a process of justification. The former is used for "discovering and exploring new ideas" and the latter is used for "justifying and proving mathematical claims" (p. 30). Duval considers reasoning for extension of knowledge, for proof, and for explanation in his theorization of teaching and learning of geometry.

We also find various approaches to reasoning surfaced in state standards: *reasoning as meaning making, reasoning used in problem solving,* and *reasoning for verification*. First, reasoning is required for making meaning, concept development, connections among concepts, and relationship building. This broad approach to reasoning seems similar to what Duval refers to reasoning for "extension of knowledge" (p. 38). Second, reasoning is used in various phases of problem solving including: 1) analyzing problem situations, 2) developing and applying strategies, 3) selecting and applying strategies and mathematical ideas, 4) explaining strategies, and 5) checking the reasonableness of the results in the problem context. This approach to reasoning could be part of what Ball and Bass refer to reasoning for inquiry. Finally, reasoning can be considered as a thought process through which students make and test conjectures, prove or disprove them, and draw conclusions. This also includes prediction, argumentation, test, justification, verification, validation, evaluation, and generalization. In this paper, reasoning pertains to mainly *reasoning for verification* as this is a more common interpretation of reasoning and more specific than the other two approaches in state standards.

Expectations of Reasoning in State Standards

Overall, it is evident that state standards acknowledge the importance of reasoning. Many state standards documents either include a reasoning standard to address reasoning expectations besides those in content strands, or explicitly state that reasoning should be incorporated throughout content strands. State standards also provide various reasoning expectations, in some cases with specific examples. While overall efforts to incorporate the significance of reasoning in state standards are observable, many state standards fail to address reasoning in a thorough and comprehensive manner. Based on our findings, here we discuss expectations we can have from reasoning expectations in state standards in order to help better promote reasoning in the classroom.

First of all, it is important that state standards explicitly address what they mean by reasoning, what aspects of reasoning are expected and why, and how such reasoning expectations could be accomplished. Our findings show that state standards rarely document this even though the importance of reasoning is addressed and that such clarification is left to readers, which causes vagueness and inconsistency of reasoning expectations. A clear notion of reasoning and a solid plan for specific expectations of reasoning are required before listing reasoning expectations in each grade and in each content strand. It will also help develop state standards in ways suggested below.

Second, state standards should address reasoning in a coherent, consistent, and connected approach. We find that reasoning expectations in many states are addressed in a fragmented manner, rather than systemically and holistically. In fact, reasoning expectations are provided in the state standards with a great variation in terms of grade level, content strand, and state. For example, primary grades have a minimal number of reasoning expectations overall (see Table 1 and Table 3). Number and operations and measurement strands include a considerably fewer number of reasoning expectations than other content strands while the strand of data analysis and probability has an extensive number of reasoning expectations (see Table 2). Grade levels in which each state document addresses reasoning expectations in content strands also vary across states (see Table 3). There does not appear to be a cohesive plan in the K-8 state standards to promote reasoning.

Such inconsistency is also noticeable when comparing various reasoning expectations across states. Expectations pertaining to prediction (e.g., "predict the results of putting together or taking apart two-dimensional and three-dimensional shapes") are the most prevalent among the reasoning expectations, followed by expectations pertaining to justification, while making arguments, proving or disproving, and using counterexamples to refute claims have less attention in the state standards.

There is also a discrepancy between components of state standards documents when addressing reasoning aspects. For example, some states have sections delineated as 'benchmarks' and 'performance indicators' to address grade level expectations. Some of the benchmark statements do not specify reasoning aspects, but their corresponding performance indicators support reasoning. This discrepancy is also found when comparing GLEs and their examples. There are cases that a GLE has reasoning aspects but the example promotes mainly procedure, or a GLE does not specify any reasoning aspects but its example requires reasoning approaches (for detail see Authors, 2006).

In order to support reasoning in an effective way, state standards should address reasoning with a deliberate plan. To list a few, reasoning should be addressed not only in process standards, but also in content strands; reasoning GLEs and their sub-GLEs should be coherent; and examples should be aligned with reasoning expectations when they are used. In addition, reasoning GLEs in the state curriculum standards should have consistency across grades and content areas. Our findings show that even in states with explicit reasoning GLEs, a particular GLE does not appear across grades and content strands. For example, an important expectation such as 'develop arguments' is provided only in one or two grades in one content strand in most states. To promote reasoning in all grades and throughout various content areas, it is suggested that essential reasoning GLEs be provided in a consistent manner.

Connections among reasoning GLEs in the state standards should also be considered. An isolated reasoning aspect alone is not sufficient to promote a deeper level of reasoning. Reasoning GLEs should be presented along with other reasoning GLEs that are related to them. In fact, there are state curriculum standards that provide 'develop arguments' without 'evaluate arguments' or 'justify arguments' and vice versa. In other states, while making predictions appears often, testing, evaluating, or justifying predictions are very rare. In order to provide systemic reasoning GLEs, multiple aspects of reasoning, as those discussed in this paper, should be considered and these aspects should be addressed in relation to others.

TABLE 2: ***Number of state curriculum documents that include GLEs in each component of reasoning by content strand***

	Number	Algebra	Geometry	Measurement	Data/Prob
Prediction	10	19	21	4	32
Generalization	1	21	3	0	2
Verification	12	7	12	2	5
Justification	15	14	13	8	20
Conclusion/Inference	0	7	3	0	26
Making Conjecture	2	0	10	0	17
Testing	4	0	10	2	15
Making Argument	0	0	6	1	10
Evaluation	2	1	2	2	17

Third, reasoning GLEs need to be clear and specific. The Council of Basic Education describes specificity as "language that describes what is the most essential for students to learn using sufficient detail to convey what is expected without dictating instructional strategies" as well as "an aspect of rigor" (Joftus & Berman, 1998, p. 19). The American Federation of Teachers (2003) also suggests that state standards "must be clear and specific enough" for related personnel to understand and to lead a core curriculum.

TABLE 3: *Grade levels in which each state curriculum document addresses components of reasoning in content strands*

	Prediction	Generalization	Verification	Justification	Conclusion /inference	Making Conjecture	Testing	Making Argument	Evaluation
AL	5, 8	-	8	2, 8	-	-	-	8	-
AK	3-5, 7-8	-	-	3-8	4-8	-	-	-	4-8
AZ	1-8	-	4-8	-	-	-	-	-	2-3
AR	1-8	6-8	5-8	3-8	1-8	-	4	5-8	5-6
CO^	3-8	-	-	3-8	3-8	3-8	3-8	5-8	5-8
DD*	K-7	2, 4, 6-8	3-4, 8	2-4, 6-8	2, 6	7	2, 7	8	8
DC*	K-5, 7-8	5-8	1-5, 8	-	1-2, 4, 6-7	7	4, 7	-	7
FL	K-8	K, 2-5, 7	K, 8	4-5, 8	7-8	6-8	4-5, 8	-	8
GA	6	-	2-3	5-6	7	-	-	-	-
HI	2-5	2-3, 7-8	5	2-8	2-5, 7-8	4-5	2-5	4-5	5-8
ID	K-8	-	-	3-8	-	4-5	3-4	-	-
IN	1-2, 6-8	-	5-6, 8	5-8	-	-	7	-	8
KS	1-8	K-4, 6-8	-	6-7	2-8	-	-	5-8	8
LA	1, 4-6, 8	8	-	8	-	-	4-6	-	-
MI	1-2	-	8	5-8	-	-	-	-	-
MN	2-3, 6, 8	-	-	-	5-6	-	-	-	-
MS	1, 3, 6-8	-	-	2	-	-	-	2	-
MO	3-5	5, 8	-	3-8	4-5	6-8	-	-	-
NV	3, 5-7	8	K, 7-8	5	5-6, 8	-	-	-	8
NJ^	3-8	-	-	5-8	3-8	5-8	6	7-8	7-8
NM	K-8	2, 5-6, 8	5-7	2, 4-8	1-8	2, 4, 7	3-4	4-5, 7-8	5-8
NC	2, 4, 8	4-5	4, 6	8	-	5	5	-	-
OH	1-5, 7	2, 4-5, 7-8	3-4	3, 5-8	1, 3-6	8	2, 7-8	7-8	-
OK	2-4, 7-8	2-4, 6-7	4	5, 8	3, 8	-	-	-	-
OR	2-4, 6-8	2-4	8	8	3-5, 7-8	7	-	8	2-8
SC	2, 5, 7-8	5, 7	-	2, 5-8	3-4, 7-8	3-5, 8	3-5, 8	5, 8	2
SD	2, 5-8	-	-	4-8	3, 6	-	6-8	-	-
TN	1-8	4-8	-	1-5	5-8	5-8	5-6, 8	-	5-8
TX	K-3, 5-6, 8	5	4	7-8	1-2, 7-8	-	-	7	8
UT^	K, 2-7	6	7	K, 5	5-6	7	4	-	6
VT	3-8	6-8	-	3-8	K-8	6-8	K-8	-	-
VA	K-2, 4-5, 7-8	7	3, 8	5-6	7-8	7	4-5,7	-	-
WA	2, 5-8	-	6-8	2-8	2	8	5,7	-	6,8
WV	1-2, 4, 8	-	5, 8	-	5, 8	-	-	8	7
WY	3-5, 7-8	4-6	-	3-8	-	7-8	-	-	-

* DD stands for Department of Defense Education Agency; DC stands for the District of Columbia.

^ Colorado and New Jersey do not have GLEs for kindergarten through grade 2 and for kindergarten through grade 1, respectively; Utah does not have GLEs for Grade 8 only.

We find that sometimes it is not clear what a particular reasoning GLE in the state standards requires students to do. Various levels of specificity and clarity are evident in the reasoning GLEs of the state standards. Some GLEs are very specific and simple; others are simple but vague. "Predict which of two events is more likely to occur if an experiment is repeated" (Virginia, grade 2, data analysis and probability) is an example of the former while "Analyze and interpret data (prediction, inference, conclusion, etc.)" (Arkansas, grade 4, data analysis and probability) is an example of the latter. The second GLE is too broad and general, not specific enough to know what is required of students even though it is addressed in a specific content strand.

In general, lack of clarity and specificity of GLEs increases the difficulty that teachers may have when interpreting and incorporating those GLEs in the classroom. In particular, when GLEs include reasoning, the quality of reasoning students engage in will be influenced by how teachers interpret GLEs, how they enact GLEs, how comfortable they are with reasoning as well as how they promote and incorporate reasoning in the classroom. Ambiguous expectations may also cause teachers' reluctance to encourage reasoning through their mathematics teaching.

Fourth, reasoning GLEs need to be integrated in content strands. It is not likely that teachers incorporate reasoning GLEs that are not explicitly connected to content areas because it is quite challenging to implement such GLEs in the teaching of a specific topic. Moreover, in this circumstance such reasoning GLEs are not likely to be assessed on state assessments. Our overall findings indicate that state standards have difficulty integrating reasoning in their GLEs. In particular, state standards with a separate reasoning section are not likely to specify reasoning GLEs in content strands. In this case, reasoning GLEs tend to be broad and general, and isolated from specific content, such as "formulate conjectures and discuss why they must be or seem to be true." Since such GLEs are not content-specific, it may be difficult to incorporate them when teaching a particular content and topic at the classroom level. Therefore, it is suggested that state standards embed reasoning GLEs in the content strands. This will increase the clarity and specificity of GLEs as well.

Additional Issues to Consider

In addition to the expectations that we can have from state standards, there are also some other issues that need to be considered in order to incorporate the reasoning expectations at the classroom level and to change classroom practices with regard to reasoning. We describe three of those issues below.

First, to promote mathematical reasoning in the classroom, appropriate assessment tools are required. It is noted throughout the examination of the state standards that reasoning expectations are not prevalent in many states. It is surmised that one plausible explanation for this is the difficulty and expense entailed in assessing reasoning. Reasoning statements are not considered correct or incorrect, rather these responses are evaluated based on the student's ability to defend or refute their thinking with plausible arguments. Assessing reasoning requires a teacher's in-depth knowledge and understanding of the mathematical concepts. Additionally, for the most part, state assessments are typically multiple-choice items. Not only is it hard to construct items to assess student reasoning, but also it takes time, personnel, and a greater cost to score. In other words, it is not easy to measure reasoning in a large-scale assessment. Assessment tools and programs at the local and state levels should be designed to incorporate reasoning aspects as stated in state standards.

Second, reasoning should be considered one of the aspects of a student's learning progress. Historically, schools rarely communicate students' progress in reasoning to parents. Teachers need to make a commitment to not only assess reasoning in the classroom, but also communicate students' growth in the area of mathematical reasoning. School culture also needs to embrace reasoning as an essential component of mathematics education and progress.

Third, in order to promote mathematical reasoning comprehensively across grades, suitable teacher training is necessary. Classrooms in general do not pursue reasoning components of mathematics in a desired way (Stigler, Conzales, Kawanaka, Knoll, & Serrano, 1999; Stigler & Hiebert, 1999). Various aspects of reasoning and their relationships in particular are still relatively foreign to teachers. It requires teachers to devote time to create and reflect on carefully planned questions and follow-up prompting of ideas. In addition, allowing students the opportunity to share and discuss their thinking pertaining to a particular problem takes time and effort, which should not be dismissed as a trivial task for classroom teachers. Maintaining a level of dedication to this process requires commitment, experience, and focused and sustained professional development.

For example, some of the state curriculum standards include GLEs, such as "explains the solution strategy," which may or may not prompt reasoning and justification. These expectations have a potential to encourage students to reason and justify their thinking, but teachers may concentrate exclusively on the procedure when students are asked to explain solutions. With such GLEs, teachers' understanding of reasoning and their questioning skills will greatly influence the width and depth of student reasoning.

Conclusion

The mathematics education community has tried to improve classroom practices that influence the quality of student learning (RAND Mathematics Study Panel, 2003). There are many ways to accomplish such a goal, one of which will be establishing more clear and comprehensive sets of state standards. We believe that state standards will significantly influence classroom practices in terms of reasoning if they provide plausible sets of reasoning expectations that are coherent, clear, specific to content, and assessable, and if teachers are appropriately supported as they implement those reasoning expectations.

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Integrating NRC Principles and the NCTM Process Standards to Form a Class Learning Path Model That Individualizes Within Whole-Class Activities

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ABSTRACT:
This paper integrates principles from two recent National Research Council Reports (How Students Learn and Adding It Up) *with the NCTM Process Standards to form a Class Learning Path Model of classroom mathematics teaching that can help teachers achieve equity in mathematics learning by assisting all students to move forward within their own learning path to at least one general, mathematically-desirable, and accessible method. This model enables leaders to integrate research results from the national reports within a single equity perspective that can be used by teachers to individualize within whole-class activities. This model consists of three parts: three continuing teaching tasks that build a Year-Long Nurturing Meaning-Making Math Talk Community that enables students to move from and relate their entering informal math knowledge to formal academic math knowledge, four Classroom Learning Zone Teaching Phases used for each math topic to move all students along their own learning path, and Inquiry Learning Path Teaching that consists of seven responsive means of assistance that facilitate learning and teaching by all.*

Two recent National Research Council Reports, *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) and *How Students Learn* (Donovan & Bransford, 2005; Fuson, Bransford, & Kalchman, 2005) identified principles that summarize research about mathematics teaching and learning. The NCTM Process Standards likewise describe vital aspects of successful teaching and learning. It would be helpful for teachers and for leaders if all of these were integrated within a single framework. That is the task of this paper. We describe a Class Learning Path Model that can help teachers to achieve equity by assisting all students to move forward within their own learning path to general, mathematically-desirable, and accessible methods. This model consists of three parts: a Year-Long Nurturing Meaning-Making Math Talk Community built by the teacher via three continuing teaching tasks, four Classroom Learning Zone Teaching Phases used for each math topic, and Inquiry Learning Path Teaching that consists of seven responsive means of assistance that facilitate learning and teaching by all (see Table 1). At the Table 1 level only principles from the two NRC reports are involved. But at the more detailed levels described later in the paper, the NCTM Process Standards are included.

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TABLE 1: *Principles and Standards in Action in the Class Learning Path Model*

Part 1: Create the Year-Long Nurturing Meaning-Making Math Talk Community to achieve the overall goal: Build resourceful self-regulating problem solvers *(How Students Learn Principle 3)* by continually intertwining the 5 strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition *(Adding It Up)*

Part 2: For each math topic, use four Class Learning Zone Teaching Phases

Phase 1: Teacher draws out and works with the preexisting understandings that their students bring with them *(How Students Learn Principle 1)*

Phase 2: Teacher helps students move through learning paths and build networks of knowledge in various math domains *(How Students Learn Principle 2)*

Phase 3: Teacher helps students gain fluency with desired method(s) so everyone moves along their learning path; individual students stop using visual supports whenever they are able to do so; fluency includes being able to explain the method; practice is kneading knowledge, so reflection and explaining still continue *(Adding It Up: fluency & understanding)*

Phase 4: Teacher facilitates remembering by occasional practice with feedback and occasional discussions to relate ideas or method(s) to new topics that might relate or interfere

Part 3: Use Inquiry Learning-Path Teaching: This consists of seven Responsive Means of Assistance that facilitate learning and teaching by all:

- Engaging and Involving
- Managing
- Coaching
    - modeling
    - cognitive structuring and clarifying
    - instructing/explaining
    - questioning
    - giving feedback

These vary by phase and over the year. Students and the teacher give assistance.

In the first author's work on the NRC reports and on the CMW Research Project, a continuing focus was on balancing the extremes of the polar positions in the "Math Wars" concerning traditional and reform teaching. This is represented in all three parts of the model. The Nurturing Meaning-Making Math Talk Community relates students' initial knowledge and experiences to the formal math vocabulary, concepts, and methods. It nurtures and supports but also consistently communicates high expectations for all: all students work hard and move along their learning path. The four Class Learning Zone Teaching Phases allow student thinking to surface and be supported within the classroom but also introduce mathematically-desirable methods that students can understand and do. Students do not jump from Concrete and Slow informal methods to rote formal Current Common methods as in traditional teaching but to methods they can relate to visual supports and come to explain as well as carry out. No one continues concrete and slow or incorrect methods as in some approaches. Inquiry Learning Path Teaching also is balanced because it clarifies that teachers must do a great deal of assisting, but that students also assist. Inquiry is in the title to emphasize that the whole learning path environment is one of inquiry: all students and the teacher are continually seeking to increase their own understandings, which sometimes occurs by helping others or by listening to the Math Talk as well as by participating in it. Inquiry does not have to mean that students must be stuck only with the methods they invent. They can be helped to more-advanced methods that they can understand with the help of the meaning-making supports and the explanations of classmates.

The Class Learning Path Model is drawn from two models developed within the Children's Math Worlds Classroom Research Project. This project worked over 12 years in a wide range of Kindergarten through Grade 5 classrooms seeking balanced approaches to teaching and learning that would work in all classrooms. The classrooms included Spanish-speaking classrooms, English-speaking classrooms, classrooms with English language learners from many backgrounds, and classrooms with a variety of inclusion students with various special needs. Many of the classrooms had 30 to 37 students in them, even in the lower grades. Thus, the model applies well to the highly challenging situations that are unfortunately too typical today but also to suburban settings with smaller classes and more homogeneous students, which were also involved in the Children's Math Worlds Classroom Research Project. The model is also consistent with the results of research in urban low-achieving schools and intervention studies with a range of students (e.g., some of these are summarized in Fuson, 2003, pp. 88-90). Part 1 of the Class Learning Path Model is adapted from part of the Mathematics Equity Pedagogy (Fuson et al., 2000), and Parts 2 and 3 are extensions of the ZPD Mathematical Proficiency Model (Murata & Fuson, 2006). The ZPD Mathematical Proficiency Model draws from several aspects of Tharp and Gallimore's (1988) Vygotskiian perspective on literacy developed in working with many children from native Hawaiian backgrounds and with other kinds of English language learners from several different cultures. Therefore, core parts of the Class Learning Path Model apply to literacy as well as to math teaching.

The Class Learning Path Model uses several concepts from Vygotsky, who theorized about how the formal knowledge of a culture was passed on to new generations both in formal and in informal teaching. These concepts will be discussed as the relevant parts of the model are described. The model uses a constructivist view of learning: students and teachers each construct individual knowledge based on their own individual life experiences, though often through interactions and assisted by a more knowledgeable person.

The Class Learning Path Model describes processes and supports that allow teachers to individualize within whole-class activities. It is easy to describe ways to individualize instruction by breaking the class apart in various ways. However, these all require management skill, time, and energy as well as special individualized materials, and this approach may decrease student's productive learning time. Our model permits continual meeting of individual needs within whole-class instruction, minimizing the need for separate specialized activities. We close this paper by relating the Class Learning Path Model to the LATCH model for integrating math instruction for English learners (Garrison, Amaral, Ponce, 2006).

Teaching real students in classrooms is a highly complex task. Our Class Learning Path Model is thus also necessarily complex. Because of space limitations and to maximize the usefulness of the presentation to leaders working with teachers, the model is primarily presented in a series of tables that can be used with teachers. The text of the paper will serve to provide background and orientation to the tables. Mathematical examples are given after Part 2 is described.

THE CLASS LEARNING PATH MODEL

Part 1

Part 1 of the Class Learning Path Model identifies three continuing teaching tasks that must be carried out all year to build and maintain the classroom environment, the Year-Long Nurturing Meaning-Making Math Talk Community, within which learning by all can flourish (see the top of Table 2). The type of learning specified as desired by both NRC reports is integrated into one overall goal stated at the top of Part 1 of Table 2. The three continuing teaching tasks come from the NCTM Process Standards and *How Students Learn* Principle 1 (see Table 2). The Teaching Tasks 1, 2, and 3 are further described in Table 3, which shows how the special classroom environment created by the on-going teaching tasks enables all learners to relate their informal initial knowledge, what Vygotsky called *spontaneous concepts* that are formed in the real world informally and without explicit teaching, to the formal academic mathematical knowledge, what Vygotsky called *scientific concepts* that are structured and hierarchical and are formed in schools or other intentional teaching situations so that students become consciously aware of them and can reflect on them. This Part 1 environment includes a safe and nurturing teaching-learning community (Teaching Task 1), coherent learning support means of assistance to help everyone build meanings for the formal constructs that relate to but extend students' entering knowledge (Teaching Task 2), and a collaborative Math Talk culture that enables students to share and discuss their present understandings and to advance their understanding by input from the teacher and their classmates (Teaching Task 3). Students' informal preexisting

TABLE 2: *Principles and Standards in Action in the Class Learning Path Model*

Part 1: Use the continuing Teaching Tasks 1, 2, 3 to create the Year-Long Nurturing Meaning-Making Math Talk Community to achieve the overall high-level goal for all: Build resourceful self-regulating problem solvers *(How Students Learn Principle 3)* by continually intertwining the 5 strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition *(Adding It Up)*

Teaching Task 1: Teacher builds the nurturing teaching-learning community (*How Students Learn Principle 1* and *NCTM Process Standard: Communication*)

Teaching Task 2: Teacher creates a cognitively supportive referential meaning-focused classroom by using coherent visual, sensory-motor, linguistic, and situation supports along with math modeling to create interest and accessibility of ideas *(NCTM Process Standards: Connections & Representation)*

Teaching Task 3: Teacher develops a collaborative Math Talk (instructional conversation) culture *(NCTM Process Standards: Problem Solving, Reasoning & Proof, Communication)*

Part 2: For each math topic, use four Class Learning Zone Teaching Phases

Phase 1: Teacher draws out and works with the preexisting understandings that their students bring with them *(How Students Learn Principle 1)*

a. Teacher elicits, values, and discusses student ideas and student methods
b. Teacher identifies students who use different levels of solution methods and those who are doing typical errors and ensures that these are seen and discussed by the class

Phase 2: Teacher helps students move through learning paths and build networks of knowledge in various math domains *(How Students Learn Principle 2)*

a. Teacher focuses on or introduces mathematically-desirable and accessible method(s)
b. Erroneous methods are analyzed and repaired with explanations
c. Advantages and disadvantages of various methods including the Current Common method are discussed so that central mathematical aspects of the topic become explicit
d. Explanations of methods and of mathematical issues continue to use quantity and/or spatial language and visual supports to help all students build networks of knowledge and move along their own learning path

Phase 3: Teacher helps students gain fluency with desired method(s) so everyone moves along their learning path; individual students stop using visual supports whenever they are able to do so; fluency includes being able to explain the method; practice is kneading knowledge, so reflection and explaining still continue *(Adding It Up: fluency & understanding)*

Phase 4: Teacher facilitates remembering by occasional practice with feedback and occasional discussions to relate ideas or method(s) to new topics that might relate or interfere

Part 3: Use Inquiry Learning-Path Teaching: This consists of seven Responsive Means of Assistance that facilitate learning and teaching by all: Engaging and Involving, Managing, Coaching (modeling, cognitive structuring and clarifying, instructing/explaining, questioning, giving feedback). These vary by phase and over the year. Students and the teacher give assistance.

vocabulary, ideas, and methods form the foundation from which the teacher builds up to the higher formal mathematical vocabulary, ideas, and methods using the resources in the Nurturing Meaning-Making Math Talk Community. There is an on-going interaction between the formal and informal vocabulary, ideas, and methods (indicated by the vertical bi-directional arrow in Table 3). All three continuing Teaching Tasks involve what Vygotsky called semiotic tools: oral language, written notations and drawings, and +physical objects that facilitate student learning of concepts and their relating of formal and informal versions of these.

Levels in Math Talk that move from traditional teacher-focused talk to student-to-student talk with teacher assistance are described in Hufferd-Ackles, Fuson, & Sherin (2004); the higher Math Talk Levels 2 and 3 give space in the classroom discourse for all voices to emerge and to move forward correcting errors and increasing understanding.

TABLE 3: ***Use the Continuing Teaching Tasks 1, 2, 3 to Create the Year-Long Nurturing Meaning-Making Math Talk Community as the Environment to Relate Students' Vygotskiian Informal Knowing to Formal Mathematical Knowing***

Formal mathematical vocabulary, ideas, and methods: Bring students up to the higher mathematics in meaningful ways and by small supported coherent steps

↕

Via a Nurturing Meaning-Making Math Talk Community

Teaching Task 1: Teacher builds the nurturing teaching-learning community: Co-creates an inclusive and participatory classroom culture in which the class co-constructs emerging related understandings for all by providing multiple levels of access (everyone can participate) through mathematizing (seeing the math in children's worlds); making math drawings; using rich language by validating all children's language and experiences while connecting them to standard language and symbols; and facilitating listening, speaking, writing, and helping competencies to make problems accessible to all

Teaching Task 2: Teacher creates a cognitively supportive meaningmaking classroom by using coherent visual, sensory-motor, linguistic, and situation learning supports along with math modeling to create interest and accessibility of ideas: Mathematical words and symbols are linked to coherent meaningful referents by mathematizing known contexts or by providing new experiences to be mathematized; rich language use by all (see Teaching Task 1); everyone makes Math Drawings or uses other visual or sensory-motor supports to facilitate reflection, discussion, analysis, and understanding of everyone's thinking

Teaching Task 3: Teacher develops a collaborative Math Talk (instructional conversation) culture of understanding, explaining, questioning, justifying, and helping that elicits, values, and discusses student ideas and methods while relating visual quantities to steps in each method and discussing mathematical attributes of methods; talkers and listeners can understand each other because Math Talk connects to referents (see Teaching Task 2); all teachers are learners and all learners (students) are teachers of themselves and of others (peer helping); all participants help to develop coherent networks of knowledge by relating ideas and experiences within instructional conversations (Math Talk)

Informal preexisting vocabulary, ideas, and methods: Start where students are and keep learning meaningful

Note. The vertical arrow indicates that the formal and informal vocabulary, ideas, and methods continually relate to each other via the Teaching Tasks 1, 2, 3.

Table 4 shows an abbreviated version of the table in Hufferd-Ackles et al. with a full description of the highest level. The term *instructional conversation* was used by Tharp and Gallimore (1988) to emphasize that the talk has learning purposes and should move participants (including the teacher) forward in their own learning paths and that it is not a teacher lecture. We included the term to emphasize that Math Talk involves students but is led by the teacher toward mathematical learning goals.

Part 2

Part 2 of the Class Learning Path Model appears in the middle of Table 2. The four Class Learning Zone Phases reflect Vygotsky's cultural model of teaching in which assistance from others, and language and actions during such assistance, eventually became assistance provided by the self, first externally and then, especially with language, are internalized into internal speech. This movement from other- to self-assistance occurred within what Vygotsky called the Zone of Proximal Development (ZPD): *the distance between the actual developmental level as determined by individual problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers.* (Vygotsky, 1978, p. 86). Each child has an individual Zone of Proximal Development (ZPD) for each kind of learning topic. However, when the four Class Learning Zone Teaching Phases are carried out within the Year-Long Nurturing Meaning-Making Math Talk Community, the whole class is working within a Class

Learning Zone and creating a Class Learning Path within which a limited number of solutions methods ranging from concrete and slow to advanced are described by students (and sometimes by the teacher) and discussed and related to each other. Within this Class Learning Zone everyone moves forward on their own individual learning path within their own zone of proximal development. Because these individual learning paths are related mathematically, Math Talk about different related methods can help everyone progress.

The first two phases in Part 2 (see Table 2) come from *How Students Learn Principles 1 and 2*. They describe how the teacher begins by eliciting student thinking and then begins to move along a Class Learning Path by focusing on or introducing mathematically-desirable and accessible methods and analyzing and repairing erroneous methods. Phase 3 comes from the *Adding It Up* focus on both understanding and fluency. Visual supports for understanding are dropped when an individual student no longer needs them, but fluency includes being able to explain a method and relate it to a visual or situational support. Phase 4 comes from the ZPD Mathematical Proficiency model (Murata and Fuson, 2006) as well as from basic learning research that indicates that occasional practice with feedback is required for a long period of time for new learning to be remembered effectively. The relational nature of mathematics also means that new related topics will arise that provide opportunities to re-view the original topic by relating it to the new topic.

Both NRC reports summarized and drew upon for their principles the explosion of worldwide research about student thinking in various math topics. This research indicates that students will be able to discuss their own ideas about a math topic that is presented in some meaning-making context or with some learning support. Therefore, Phase 1 is fruitful (will lead to student ideas and methods) if it occurs within a Nurturant Meaning-Making Math Talk Community (Part 1 of the model). This same research indicated how student methods for many topics fall into a learning path of increasing abstractness and abbreviation that move from concrete and slow methods to faster methods, some of which are general and accessible. Research from around the world also indicates that different solution methods are taught in different countries. These methods are often complex and abbreviated and thus relatively difficult to learn with meaning. In the United States these are often called "the standard algorithms," but *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) stressed that this term is misleading because different algorithms have been taught at different times in this country. We therefore call these methods the Current Common methods. Research has identified instead algorithms and other kinds of solution methods that are mathematically-desirable and more accessible (MD & A) to students than are the Current Common methods. These more accessible methods fit students' thinking better, so they are easier for students to understand and to explain. Most are easier to do procedurally and are less prone to errors than are the Current Common methods. But each clearly uses at least one important mathematical idea and so is a worthy focus of Math Talk that will make this idea clear to students. Some of these methods are described in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001), in the research volume accompanying the NCTM Standards 2000 (Fuson, 2003), and in Fuson (2006).

These mathematically-desirable and accessible methods are what enables instruction to be differentiated within whole-class activities when they are taught with all three parts of the Class Learning Path Model. The Nurturant Meaning-Making Math Talk Community enables all children in a class to understand at least one of the mathematically-desirable and accessible methods when it is taught with the seven means of assistance that constitute the Part 3 Inquiry Learning Path Teaching (to be discussed shortly). Table 5 shows how the differentiated learning works within the four whole-class phases. Each student does advance within his/her own learning path. But what makes things manageable within the whole-class context is that, for any given math topic, there are a limited number of methods that students develop and share and there are also a limited number of kinds of errors made by students. So it is possible to share the range of student methods within the Nurturant Meaning-Making Math Talk Community and to surface and address the errors within the Math Talk. The coherent learning supports introduced for the topic enable the Math Talk to be comprehensible to all listeners.

Phase 1. In Phase 1 methods are elicited from students. These (see Table 5) include incorrect methods, concrete and slow methods, general and accessible methods, and sometimes the Current Common method which is identified in Table 5 as a student method if it is introduced by a student rather than by the teacher. This process allows all cultural methods taught at home to be voiced in the classroom, where they can be explained with the help of the

TABLE 4: ***Levels of Math-Talk Learning Community: Teacher and Student Action Trajectories***

Components of the Math-Talk Learning Community			
A. Questioning	B. Explaining math thinking	C. Source of math ideas	D. Responsibility for learning
Overview of Shift over Levels 0 - 3: The classroom community grows to support students acting in central or leading roles and shifts from a focus on answers to a focus on mathematical thinking.			
Shift from teacher as questioner to students and teacher as questioners.	Students increasingly explain and articulate their math ideas.	Shift from teacher as the source of all math ideas to students' ideas also influencing direction of lesson.	Students increasingly take responsibility for learning and evaluation of others and self. Math sense becomes the criterion for evaluation.
Level 0: Traditional teacher-directed classroom with brief answer responses from students.			
Level 1: Teacher beginning to pursue student mathematical thinking. Teacher plays central role in the math-talk community.			
Level 2: Teacher models and helps students build new roles. Some co-teaching and co-learning begins as student-to-student talk increases. Teacher physically moves to side or back of the room and directs from there.			
Level 3: Teacher as co-teacher and co-learner. Teacher monitors all that occurs and is still fully engaged.			
Teacher is ready to assist, but now in more peripheral and monitoring role (coach and assister).			
Teacher expects students to ask one another questions about their work. The teacher's questions still may guide the discourse. The teacher's questions still may guide the discourse. Student-to-student talk is student initiated, not dependent on the teacher. Students ask questions of each other and listen to responses. Many questions are "Why?" questions that require justification from the person answering. Students repeat their own or other's questions until satisfied with answers.	*Teacher follows along closely to student descriptions of their thinking, encouraging students to make their explanations more compete; may ask probing questions to make explanations more complete. Teacher stimulates students to think more deeply about strategies.* Students describe more complete strategies; they *defend and justify* their answers with little prompting from the teacher. Students realize that other students will ask them questions, so they are motivated and careful to be thorough. Other students support with active listening.	*Teacher allows for contributions from students during her explanations; she lets students explain and "own" new strategies. (Teacher is still engaged and deciding what is important to continue exploring.) Teacher uses student ideas and methods as the basis for lessons or miniextensions.* Students contribute their ideas as the teacher or other students are teaching, confident that their ideas are valued. Students spontaneously compare and contrast and build on ideas. Student ideas form part of the content of many math lessons.	*The teacher expects students to be responsible for co-evaluation of everyone's work and thinking. She supports students as they help one another sort out misconceptions. She helps and/or follows up when needed.* Students listen to understand, then initiate clarifying other students' work and ideas for themselves and for others during whole-class discussions as well as in small group and pair work. Students assist each other in understanding and correcting errors.

meaning-making supports in the classroom and related to other methods.

Phase 2a. In Phase 2a the mathematically-desirable and accessible methods are introduced by the teacher or by the math program (e.g., as methods used by characters in a story or students in someone's class), again linked to the visual and other meaning-making learning supports. Because of the visual and verbal learning supports in the Math Talk Community, one of these methods can be learned by each student in the class, in contrast to the current common method, which is more complex and abstract. We included two mathematically-desirable and accessible methods in Table 5 because research has identified in many areas two such methods that vary in the mathematical attributes they emphasize. Introducing both

TABLE 5: *Differentiated Learning Within the Class Learning Zone Phases: Everyone Advances Within Own Learning Path*

Phase 1: Students enter with a range of methods ranging from concrete and slow to advanced and rapid; some may know the current common method [CC], which is labeled student method A below if it is demonstrated initially by a student.

Phase 2a: Teacher focuses on or introduces mathematically-desirable and accessible method(s) [MD&A] and ensures that erroneous methods are analyzed and repaired with explanations.

Phase 2b: Teacher introduces current common method [CC] if it has not already been demonstrated by students, and students relate it to MD&A method(s) during Math Talk.

Phase 3 & 4: Students become fluent in one mathematically-desirable and accessible, general and accessible, or current common method; many students become fluent in two or three such methods. Students maintain or finally achieve fluency by occasional practice with feedback and occasional discussions to relate ideas or method(s) to new topics that might relate or interfere.

Type of Method	Phase 1	Phase 2a	Phase 2b	Phases 3 & 4
Current Common [CC]	student method A?	student method A?	**CC method related to MD&A methods**	CC method?
Mathematically-Desirable & Accessible [MD&A]	student method B?	**MD&A method a** **MD&A method b**	MD&A method a MD&A method b (may be abbreviated)	MD&A method a MD&A method b (may be abbreviated)
General & Accessible	student method C	may continue	may continue	may continue
Concrete & Slow	student method D	**move on to MD&A method a or b**		
Incorrect	student method E	**Discuss and repair errors in methods**	**Monitor; repair if reappear**	**Monitor; repair if reappear**
Number of methods by one student	most 1 or 0	most 1, some 2 or 3	all 1 MD&A or G&A or CC many 2 or 3 methods	all 1 MD&A or G&A or CC many 2 or 3 methods

Note. ? means that this method may not be used by any student. More than one student method of a given type may be used.

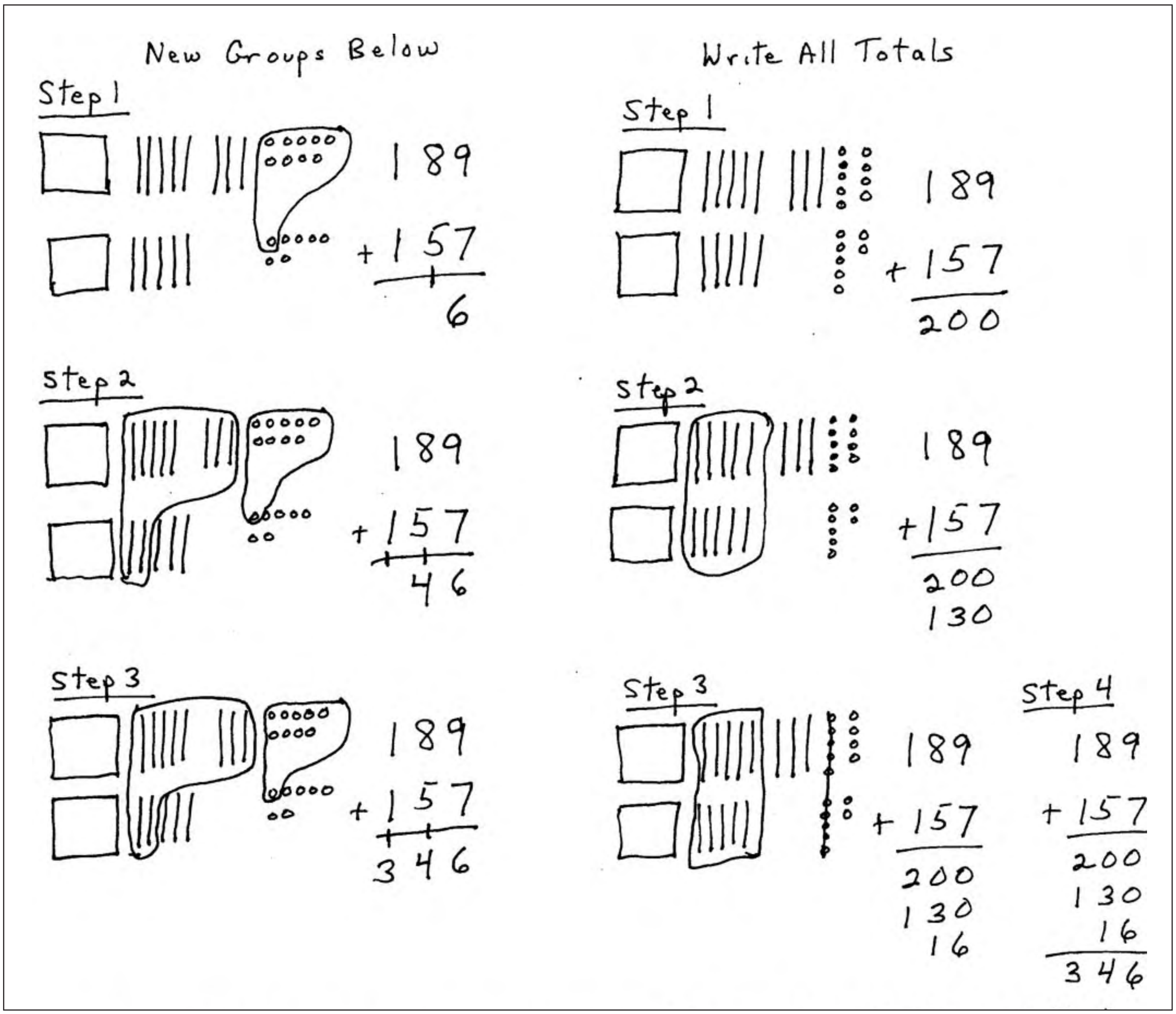
permits fuller understanding of the math topic even for those students who learn only one of the methods. They may vary in abstractness so that less-advanced students choose the more concrete or visual method, or they may just appeal to individual differences in students (Fuson, 2006). In all explanations in all phases, it is important to link the math drawing or other visual support to the formal math method for each step of that method. It is such tight linking that enables the meanings for the visual or contextual supports to become attached to the formal math method and notations and thus to take on those meanings.

During Phase 2a all students experience and discuss advantages and disadvantages of the mathematically-desirable and accessible methods. Students who were using concrete and slow methods are asked to choose one of the methods, become fluent in it, and become able to explain its steps using meaningful standard mathematical language. Such explanations are given first by more-advanced students and clarified and extended as needed by the teacher, so these less-advanced students hear examples before they explain themselves. Students who were using a general and accessible method (or a Current Common method) may continue using their method as long as they can explain it linked to the visual quantity support being used for that topic; the teacher and other students assist with such explanations. Errors also continue to be discussed and repaired. By the end of Phase 2a almost all students have moved from concrete and slow and from incorrect methods to a mathematically-desirable and accessible method. Some students enjoy trying all of the methods that have been introduced (all those beyond the concrete and slow methods) and may vary the method they use for different problems.

Phase 2b. In Phase 2b the Current Common method is introduced by the teacher if it has not already been demonstrated by a student and is related within Math Talk to the mathematically-desirable and accessible methods. Such methods are chosen to relate easily to the Current Common method so that Math Talk is accessible and so that parents who know the Current Common method can readily understand the mathematically-desirable and accessible methods.

Phase 3. In Phase 3 students build fluency for their chosen method or methods. No student is now using a concrete and slow method, and errors have been greatly reduced. Some less-advanced students may still be making math drawings, but many students no longer are. Math Talk explanations continue to enable all students to build or strengthen their network of knowledge for the topic as well as increase their fluency for their method(s). We found that in many classrooms the majority of students during Phase 3 enjoyed mastering and using two or three methods. The mathematically-desirable and accessible methods are

FIGURE 1. ***Linked drawing and numerical steps for addition methods.***



chosen so that they are rapid enough to be used for life. Thus, they do not have to be replaced by the Current Common method, though they may be whenever a student so chooses and can explain the Current Common method (this keeps the emphasis on understanding as well as on fluency).

Phase 4. Phase 4 is important because sometimes errors can creep back in, especially for older students who have been using an erroneous method for a year or more before learning a mathematically-desirable and accessible method (e.g., subtracting the smaller from the larger number even if the larger number is on the bottom is an extremely widespread error at all grades and even into high school). During this phase it is often enough to ask students to think about a math drawing (or other visual support) for them to be able to correct the error.

Examples of the Types of Methods

Multidigit addition. Examples of mathematically-desirable and accessible methods for multidigit addition are shown in Figure 1 along with math drawings of the hundreds, tens, and ones that students would make to support their understanding and explanations of their numerical methods. These methods were in *Adding It Up* Kilpatrick, Swafford, & Findell, 2001; p. 202) and are discussed more fully in Fuson (2006). The first method (New Groups Below) is just like the Current Common method (which could be called New Groups Above, see both of these methods in Figure 2) except that the new group (new ten, hundred, thousand, etc.) is written below the next left column on the line rather than above it. The New Groups Below method generalizes to any number of places and has three advantages over the Current Common (New Groups Above) method:

a) When you write the new group below, it is near the ones of the teen number you made, so you can see the whole teen number more easily. This clarifies what you are actually doing when you make the new group and put it in the next left column. For example, when adding the 9 ones and 7 ones, you can see the 16 much more easily in New Groups Below (see Figure 1) than in the New Groups above method where the 1 and the 6 are separated so far apart.

b) It is much easier to add the numbers whenever you have a new group because you add the two numbers you see in the problem (e.g., 8 tens and 5 tens in Figure 1) to get 13 tens and then add the 1 ten waiting below to get 14 tens. In the Current Common (New Groups Above) method (see Figure 2), you add the 1 to the top number 8, hold that total 9 in your mind while you add to it the bottom number 5 (you can't even see the second number 9 and you can see the old top number 8 that you are no longer using).

c) Some students object to the Current Common (New Groups Above) method, saying that you are changing the problem when you put the 1 up there. And actually you are changing the addition problem when you do that. For example, the 1 new ten above the 8 tens in Figure 2 changes the top number from 189 to 199. In New Groups Below the new 1 group stays down below in the answer space not changing the problem.

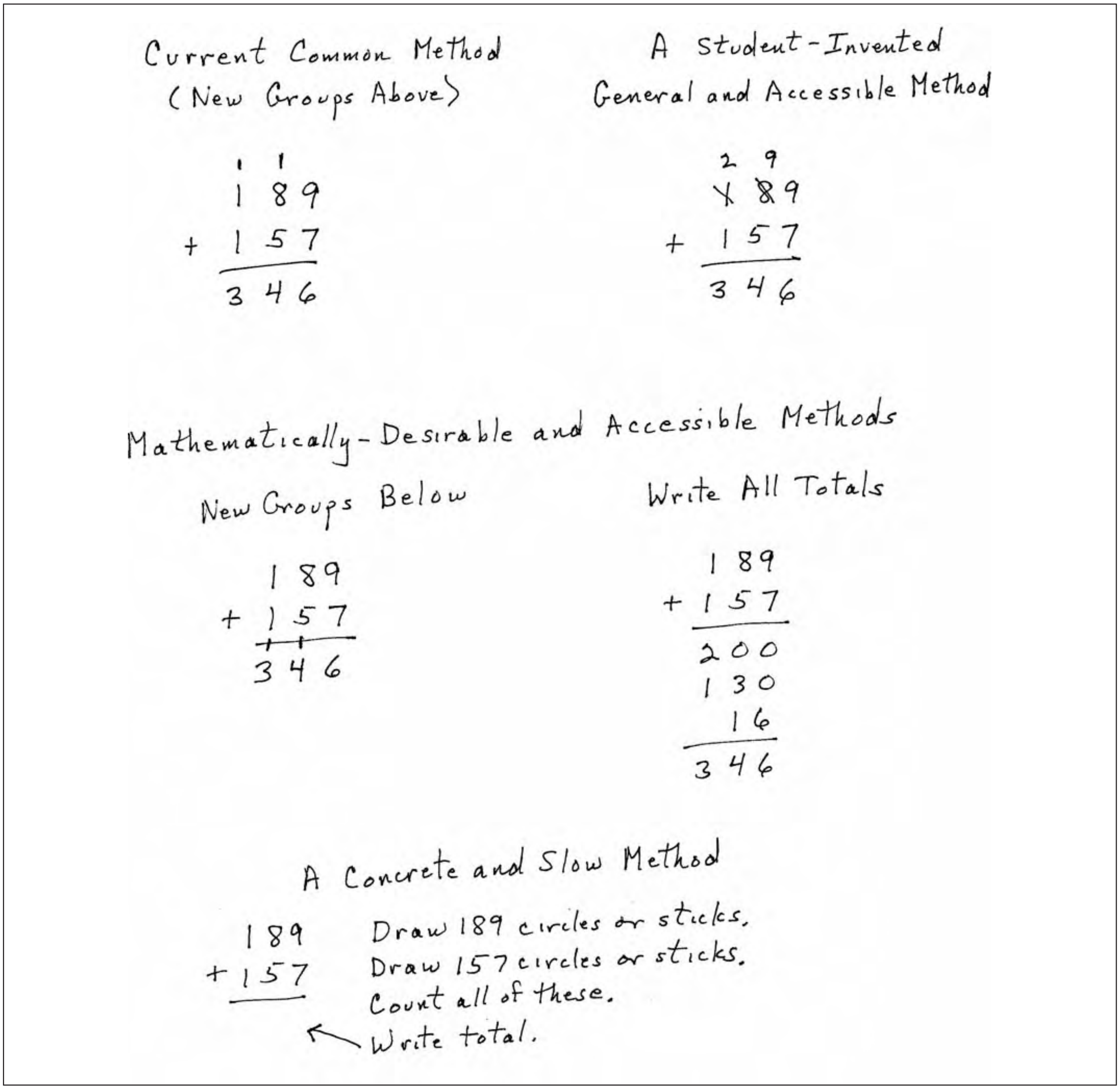
The second research-based mathematically-desirable and accessible method, the Write All Totals method (see Figure 1), shows the total of each place value written using all needed zeroes. This method can go from the left (shown in Figure 1) or from the right (the rows of subtotals would just be reversed). Most students prefer to go from the left; teachers of special needs students find this method valuable.

The Write All Totals method eventually becomes cumbersome for very large problems, but is worth introducing and discussing to help less-advanced students see the values they are adding for numbers in the millions. Seeing both New Groups Above and New Groups Below in many places helps students understand the generality of making 1 new group of the next larger multiunit from ten of the smaller units to the right. The 3 advantages of New Groups Below continue for such large numbers.

The New Groups Below and Write All Totals methods generalize to decimal positions to the right of one. The Write All Totals method helps students see how to add thousandths, hundredths, and tenths and to verbalize that 10 thousandths make 1 hundredth and 10 hundredths make 1 tenth; these are initially difficult because the verbal patterns are in the opposite direction to those for whole numbers where 10 hundreds make 1 thousand. For these places we use dimes, pennies, and a picture of a sectional tenth of a penny to help students visualize and remember that the places are getting smaller as you move to the right (they are getting one-tenth as big).

Figure 2 shows the above three methods and also a student-invented general and accessible method and a concrete and

FIGURE 2. ***Methods for multidigit addition.***



slow method. The former is a variation of the Current Common (New Groups Above) method in which the one new ten or hundred is added into the top number rather than being written above ready to add in. This method was invented by students using base-ten blocks, who added the new ten or hundred in with the blocks for the top row (Fuson & Burghardt, 2003). This method could also be done with math drawings such as shown in Figure 1. It has the same first advantage over the Current Common method as does New Groups Below: the addition for each column is easier. This method is general (it can be extended to larger whole and to decimal numbers), and it is accessible to students. In the Children's Math Worlds Project we introduced students to New Groups Below rather than this method because of the latter's three advantages and because some students confused this method with subtraction because of the crossing out of top numbers.

A concrete and slow method is to make a drawing of things (or circles or sticks) for each number and count all of the things by ones. Many students do invent such a method for 2-digit numbers; even for such numbers it is very slow and often inaccurate. Students who learn quantity drawings that show hundreds, tens, and ones such as in Figure 1 have no need to do such a slow method and can immediately understand and use one of the mathematically-desirable and accessible methods. Students may also use methods that count on by hundreds and tens; these are not general (they become very awkward even for large hundreds) and are not accessible (many students do not have those skills of counting on, and they take time to develop).

These examples show all of the types of methods listed in Table 5. The Current Common method for addition is relatively accessible to students. The Current Common methods for multidigit subtraction, multiplication, and division are less so. Introducing mathematically-desirable and accessible methods such as those shown in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) and in Fuson (2006) can be very helpful in allowing all students to move up to such a method that they can understand, do, and explain.

The importance of math drawings. In the Children's Math Worlds Classroom Research Project we found that moving as rapidly as possible in each topic to having students make math drawings along with their solution methods was extremely powerful in supporting everyone in the Math Talk Community to understand and participate in the instructional conversation. Math drawings focus on the mathematical aspects of a problem and are as simple as possible (e.g., for a word problem about cats, children draw circles rather than pictures of cats). Math drawings can be made rapidly on the class board, on class activity sheets, and on homework. They help English language and less-advanced learners follow the Math Talk. Non-English speakers can gesture to parts of their math drawing and then to their numerical or geometric solution to relate these, and a helping classmate can voice their explanation, checking with the explainer that it is correct. Many students, even native English-speakers, usually can comprehend more than they can say, but this process of explaining using a math drawing allows them to participate before they have become fluent in the formal math English needed for a full explanation.

One can see the power of math drawings by looking at the quantity math drawings shown in Figure 1. The drawings themselves were taught as part of Teaching Task 2 when students were discussing place value. The vertical ten-sticks were originally ten circles connected by a vertical stick. The hundred-boxes originally contained 10 ten-sticks. Variations in math drawings come from individuals and are not associated with any particular numerical method. Students link the drawings step-by-step to each mathematically-desirable and accessible numerical method and explain each step in their drawing linked to that step in their numerical method. They must use quantity language (hundreds, tens, ones) when adding tens or hundreds. For example in New Groups Below, they say "eight tens plus five tens is thirteen tens plus one more ten waiting here below is fourteen tens, which is one hundred and four tens." Or they may say "eighty plus fifty is one hundred thirty (they can see in the drawing how 8 tens need 2 more tens to make 10 tens, which equal 1 hundred) plus one more ten from the ones makes one hundred forty." They do not say "eight plus five" when adding tens or hundreds. This quantity language helps the numerical method to take on these quantity meanings, which will remain when students no longer need to make the drawings. They now can make the verbal quantity explanations when looking only at the numerical method.

Students vary in how they make the new 1 ten or new 1 hundred from the ones or from the tens. Each such variation supports different mental single-digit methods, which are also facilitated by the 5-groups in the drawings. The top left drawing shows in the ones that 9 needs 1 more to make ten; when that 1 is taken from the 7 it becomes 6, so 1 ten plus 6 equals 16. The middle left drawing shows the same make-a-ten method for adding tens, but here the 8 tens need 2 tens taken from the 5 tens to make 10 tens, which leaves 3 tens. Step 3 on the right shows the 5s within the 9 and the 7 added to make 1 ten, leaving the 4 in the 9 and the 2 in the 7 to be added to make 6. So as students explain their methods and their classmates see variations in their drawings, different more-advanced mental methods for single-digit addition are also supported.

Single-digit subtraction. Methods of single-digit subtraction that move from the concrete and slow Take Away method to the mathematically-desirable and accessible Count Up and Make a Ten methods to the current common Recall/Memorize method are shown in Figure 3. *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) summarized the massive world-wide research literature on the developmental/experiential levels in single-digit addition and subtraction methods found around the world (see also Fuson, 1992, 2003). These levels move from

a) early conceptual structures in which students are able to consider only one number at a time: to Take Away for 14 – 8 = ?, they first make 14, then take away 8, then count the rest as 6; to

b) an embedded number concept in which an addend is embedded within the total: to Count Up, 14 – 8 = ? is thought of as 8 + ? = 14 and the 8 is embedded within the 14: "9, 10, 11, 12, 13, 14; that's 6 more from 8 to make 14." to

c) derived fact strategies in which students can make chunks within addends to use a known problem into an unknown problem: for Make a Ten, 10 + 4 = 14 is used to find 8 + ? = 14: 8 + 2 makes 10 plus 4 more in the 14 makes 6.

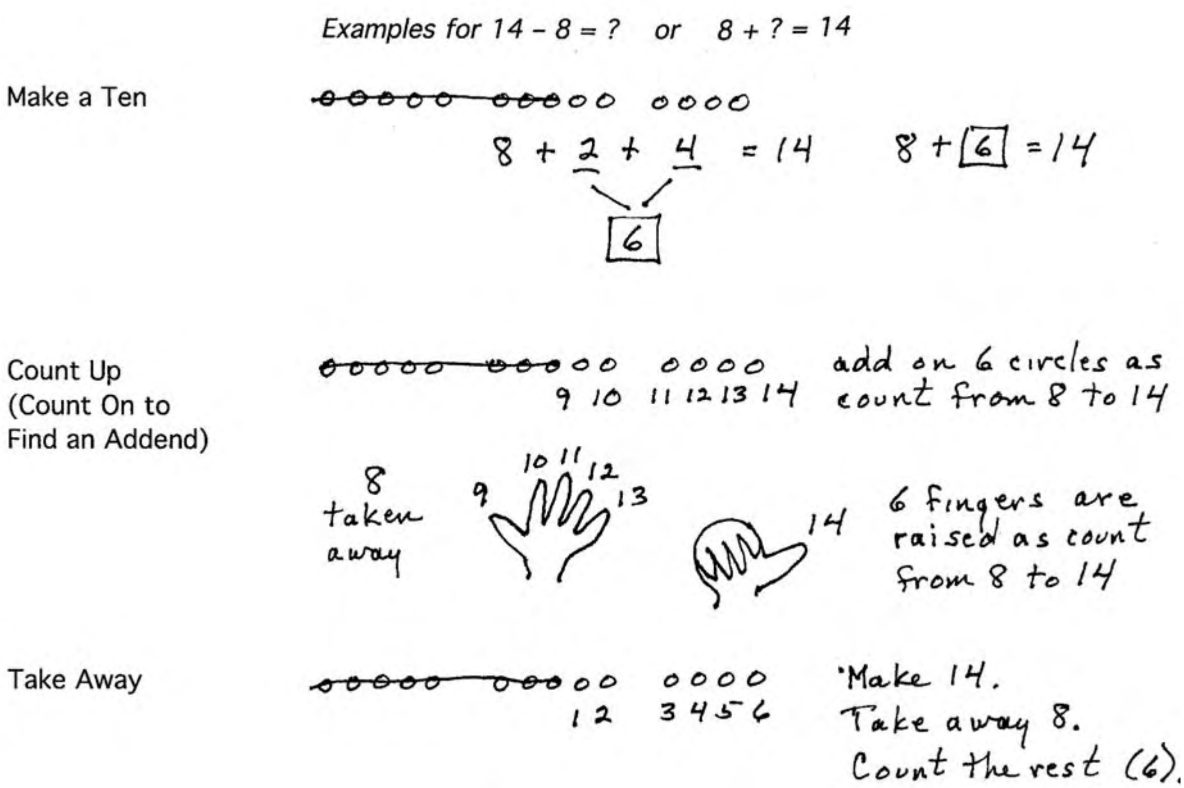
Many countries around the world help students move from the concrete and slow informal Take Away method to the mathematically-desirable count up method, and some also help students to move on to the more-advanced Make a Ten method. This method is particularly valuable in multidigit subtraction, where ungrouping gives a top number that is a ten from the next left column and the number in the top column. In Korea, the multidigit subtraction algorithm taught is to write the new ungrouped10 above the column to the right because that facilitates the make-a-ten single-digit subtraction for that column (Fuson & Kwon, 1992). For example, if 4 is in the top ones place and an 8 is below to subtract, after ungrouping a 10, a Korean child would see that 4 and a 10 above it, so could easily do Make a Ten by thinking from 8 to the 10 they see is 2 and 4 more in the 4 they see makes 6. In the U. S. it is typical to show that top number as a teen number, i.e., to write a small 1 to the left of the top number or cross out and rewrite the whole top number as 14 (we did the latter because it is clearer). In this case one can still do the Make a Ten method to find 14 – 8, but the 10 is not there visually as a support.

In the United States many students invent and use a counting down method. But these methods are difficult to carry out, and many students make errors in carrying out this method. Students in fact use four different counting down methods used, two of which are systematically wrong (Fuson, 1984). You can start counting down with the total, and then the unknown addend will be one less than the number you say when you've counted down the known addend. For example, for 14 – 8, "14, 13, 12, 11, 10, 9, 8, 7 (counted down 8, usually kept track of by raising 8 fingers as you count), so there are 6 (the next number down fom 7) left." Or you can start counting down one less than the total, and the last number counted down is the answer: "13, 12, 11, 10, 9, 8, 7, 6 (counted down 8), so there are 6 left." But students do both incorrect combinations of these, yielding an answer one too big (start with 14 and give the 8th word said as the answer: 7) or one too small (start with 13 and give the number following the 8th word said as the answer: 5).

Learning to solve subtractions by forward methods (i.e., as 8 + ? =14) has two major advantages: the forward methods make subtraction as easy as addition, and they emphasize the relationships between addition and subtraction. Addition is finding an unknown total, and subtraction is finding an unknown addend. For this reason (and to simplify terminology), in the Children's Math Worlds Research Project we distinguished adding counting on from subtracting counting on (also called counting up) by calling the adding method Counting On to Find the Total and the subtracting method Counting On to Find an Addend. The keeping track process for subtracting is easier than that for adding because you just stop when you hear the total and then look at your fingers to see the answer. For adding you must monitor your fingers until you see the second addend you are counting on. Similarly, Make a Ten to subtract is easier than Make a Ten to add because for the former, you need only find the amount to make ten with the known addend (e.g., 8 + 2 = 10) and then add that amount to the ones number you see in the teen total (add 2 to the 4 in 14). For adding Make a Ten, you need to separate the second addend into the amount to make ten (the same first step as in adding) and then find the rest of that second addend to make the ones place in the teen total: 8 + 6 is 8 + 2 + ?; think 2 + ? = 6, so 4, so 10 + 4 is 14. This is easier in East Asian languages where 14 is said as "ten four" so students do not have to know the extra European teen language step of knowing that 10 + 4 is "fourteen = 14." Very low Taiwanese students learn to use make-a-ten for subtraction before they can do so for addition (Duncan, Lee, & Fuson, 2000). In the CMW Project we found that first graders of all levels can learn Counting On to add and to subtract, and some/many also can use the Make-a-Ten methods. Others began to use the Make-a-Ten methods during multidigit addition and subtraction. Still others remained with counting on for single-digit adding and subtracting. As with all mathematically-desir-

FIGURE 3. *Methods of single-digit subtraction.*

Current Common	Recall/Memorize
Mathematically-Desirable & Accessible	Make a Ten
	Count Up (Count On to Find an Addend)
General & Accessible	none (Count Down is not accessible and is often inaccurate)
Concrete & Slow	Take Away
Incorrect	Count Down is frequently incorrect



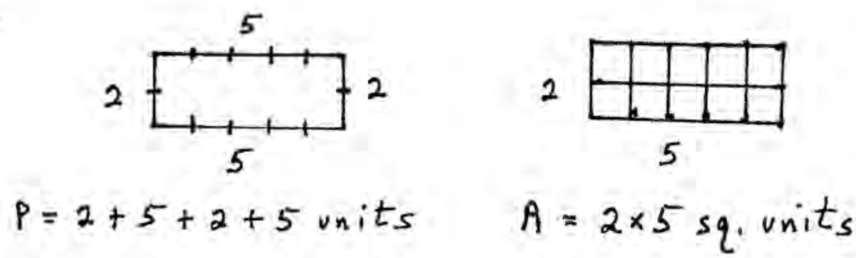
able and accessible methods, this is rapid and accurate enough to be used in any more complex problem solving and thus does not have to be replaced.

The conceptual way to enable students to move through the worldwide levels shown in Figure 3 and relate subtraction to addition is to show subtraction using 5-structures and 10-structures within math drawings such as are shown in Figure 3 and to take away the first objects rather than take away from the right. So for all three levels in Figure 3, the methods build on each other (e.g., you can see that 2 more from 8 makes 10 and there are 4 more in 14). All these methods show taking away 8, and you can even start Count Up and Make a Ten by saying "8 taken away" and continuing with the method.

FIGURE 4. ***Methods of finding the perimeter of a rectangle.***

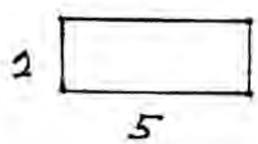
Current Common	P = 2 x (l + w)
Mathematically-Desirable & Accessible	P = 2(b + h) P = 2 x (b + h)
General & Accessible	P = 2l + 2w P = l + w + l + w
Concrete & Slow	count all length segments around the edge
Incorrect	l + w because only those numbers appear on the rectangle shown for the problem

Meaning of perimeter versus area



Misleading perimeter problem (only 2 sides shown)

Find the perimeter.
Find the area.



The current common method Recall/Memorize is of course useful for smaller additions and subtractions (most of the totals below ten). Students use this method from the very beginning (e.g., for 1 + 1) of their addition/subtraction experience, and they continue to solve new unknown totals or unknown addends by this method. But especially for totals between 10 and 18, the Make a Ten and Count Up (Count On to Find an Addend) methods are fast and accurate enough for all purposes, and their use can even reduce the interference between addition and multiplication memorized facts that interferes with multiplication learning. State standards should reflect the massive worldwide research on these levels and require students to be fast and accurate with single-digit addition and subtraction rather than specify the method by which they must demonstrate such fluency (only memorized or recalled "facts").

Perimeter of rectangles. Methods for finding the perimeter of a rectangle are shown in Figure 4. Math drawings that show the meanings of perimeter and area are shown below. Students initially need experiences drawing rectangles using inches and centimeters to experience different measure units in use in perimeter and area. Such experiences can help them see the unit lengths of inches or of centimeters so that these, rather than the endpoints, are the units that are counted to make the perimeter.

The vital visual/conceptual points to make are that perimeter is the total of the *length units* all of the way around the rectangle and area is the total of the *square units* that cover the surface of the rectangle. Because perimeter problems are typically shown with a rectangle that has numbers for only two adjacent sides, a common incorrect method for finding perimeter is to add only those two numbers shown rather than also adding in or otherwise using the other two sides. In the CMW Research Project, we found that it helped students understand both of these points if they made two small math drawings for such a problem (see the next to bottom row in Figure 4). For perimeter, they marked and labeled the length units all around the rectangle and wrote the perimeter as the sum of all four sides. For area, they drew a second rectangle, drew in the grid of square units, and wrote the area product. Students stopped making such drawings whenever they no longer needed them.

The conceptually most accessible methods for perimeter move from the concrete and slow informal method of counting all of the length units around the sides of the rectangle (the basis for understanding what perimeter is) to general and accessible numerical methods of adding the length units rather than counting them (see Figure 4). Students invent the latter methods once they understand what perimeter is. The current common method emphasizes that one only needs the lengths of adjacent sides by using the more-advanced but less-accessible formula "the sum of the length and the width taken two times." The mathematically-desirable and accessible methods are variations of this current common method that use *base* and *height* instead of *length* and *width* in order to relate rectangles to parallelograms and triangles, where *base* and *height* instead of *length* and *width* are used. This also avoids the ambiguities in the terms *length* and *width* (is the *length* the *base* or is it the longer side?). Of course, students as always need to be introduced to the current common method and to vocabulary it uses (the terms *length* and *width*).

Part 3

Based on Vygotsky's concept of the Zone of Proximal Development (ZPD) within which a learner is assisted by more knowledgeable others, Tharp and Gallimore defined teaching as follows: *Teaching can be said to occur when assistance is offered at points in the ZPD at which performance requires assistance.* (1988, p. 31). Tharp and Gallimore identified 6 means of assistance used in teaching. We identified in the ZPD Mathematical Proficiency model (Murata & Fuson, 2006) one more, resulting in the 7 means of assistance that constitute Inquiry Learning Path Teaching, Part 3 of the Class Learning Path model (see Table 1). These means of assistance are used within the Part 1 Nurturing Meaning-Making Math Talk Community and throughout the Part 2 four Class Learning Zone Teaching Phases. Both the teacher and students assist learning. A vital role of the teacher all year is to assist students in learning how to assist better, and all students can improve in such assisting. However, we found in the Children's Math Worlds Research Project that that even some first graders and kindergarten students are natural assisters without such teacher help so that the Math Talk Community has an initial basis of assistance from classmates as well as from the teacher.

There are three main categories of responsive assistance: Engaging and Involving, Managing, and Coaching. Engaging and Involving is important throughout the four phases but is especially critical at the beginning of a new topic where some students may feel overwhelmed. Managing by students of course must be set up by the teacher, but students can take over substantial aspects of managing materials and student movement if the teacher assists them to learn to do so. The five Coaching means of assistance in Table 1 are ordered from the most to least structuring done by the assister: modeling, cognitive structuring and clarifying, instructing/explaining, questioning, giving feedback.

The word *responsive* is crucial for the means of assistance. This means that assistance is only given to individuals when they need it and at the points at which they need it. Doing more creates dependence. The Math Talk Community permits assistance to be given individually, but the teacher and classmates must learn to give long wait times while students attempt to explain before jumping in to help. Our CMW teachers called this "biting their tongue" and stressed that it was initially difficult to do; they were used to doing most of the talking in the classroom. However, when they did leave space for student voices to emerge, and managed the class from the side or back of the room during Math Talk, they were frequently impressed by what their students said. The mathematical points or methods they planned for the lesson mostly would come from the students, though often in a different order than they anticipated.

At the beginning of the year, the teacher is the main assister but concentrates on supporting students to use all of the means of responsive assistance. This requires that students become close listeners, be collaborative and supporting, and be mutually adapting in their interactions. As the year continues, students provide a great deal of assistance in whole-class situations and increasingly in pairs or groups. All means of assistance are used to help students become better assisters, but the five Coaching means of assistance are especially important.

During Phase 1, students use modeling and instructing/explaining (always with possible assistance from the teacher) rather than these being used primarily by the teacher, as is traditional when introducing a new topic. During Phase 2 the teacher may need to do more modeling and instructing/explaining to ensure that the mathematically-desirable and accessible methods are clear to everyone, but in most classes much of this can also come from students especially after the classroom is functioning strongly. Instructing/explaining, questioning, and giving feedback (the other three Coaching means of assistance) occur most often in Phases 2 and 3 and are done by students and by the teacher. These Coaching means of assistance help students understand the learning supports introduced by the teacher and the math program for each topic and facilitate students' conscious formal learning of the formal math vocabulary, ideas, and methods.

The Vygotskiian move within the Zone of Proximal Development from other-assistance to self-assistance means that in Phase 3, all means of assistance are used less often, only on more-difficult aspects, and only for students who need them. Some students now may be observed using self-regulating speech while solving a problem; this speech may be similar to things their classmates or teacher said while solving. In Phase 4 the means of assistance may be needed very little.

Once the Nurturing Meaning-Making Math Talk Community and the seven Inquiry Learning Path Teaching means of assistance are well-established in the classroom, students can carry more of the responsibility, especially in whole-class Math Talk discussions. The teacher must always introduce the new learning supports for a new math topic, begin by eliciting student thinking, and be sure that the mathematically-desirable and accessible methods are introduced and discussed. But students later on in the year can manage much of the Math Talk. Many substitute teachers in our CMW Project classrooms commented later to the regular teacher that the students directed themselves during Math Talk and decided when they understood and were ready to practice alone. These substitute teachers initially did not even have the concept of students doing Math Talk, but the students could do it for topics they already knew without the support of the substitute teacher. We found that some students even in kindergarten are capable of high levels of assisting if they are given opportunities to do so and are helped to assist more productively.

COHERENCE AND BALANCE

Ways in which each part of the Class Learning Path Model is coherent and balanced were summarized at the beginning of the paper. Another crucial aspect of coherence and balance that is required for the Class Learning Path Model to work most effectively is programmatic coherence. This is necessary to provide adequate introduction and practice of prerequisites for a topic so that less-advanced students will be in a position to understand the mathematically-desirable and accessible methods. Effective functioning of the first three Class Learning Zone Phases for central grade-level goals requires sustained deep learning that takes time rather than a spiral approach where there is never enough time for moving everyone to mastering a mathematically-desirable and accessible method. Coherence in the learning supports across topics and across grades can reduce learning time and increase understanding and fluency, especially if these supports are chosen to allow students to experience various crucial mathematical ideas across the supports. Research to develop such coherence was a primary task of the Children's Math Worlds Research Project. The learning path curriculum that was developed in the project is now published by Houghton Mifflin as *Math Expressions.*

Of course not all topics can have an extensive period where all students explain their thinking. For some less-important topics, the teacher will go through the first three phases all in one day or in a couple of days, either because it is a small or less-important topic or because many students already have the necessary knowledge. Even in such abbreviated cases, students can still have opportunities to share their thinking and previous knowledge and practice saying any new math terms or relationships through choral practice or quick whole-class turn-taking routines. Pressures of time may even mean that teachers occasionally do much of the explaining on some days because it is faster. But it is vital that the focus on meaning-making supports is always maintained so that visual and contextual examples are always provided initially and related gesturally as well as verbally to the formal math notations and vocabulary. When mathematically-desirable and accessible methods are not available in research or in the math program a teacher is using, the current common method in the program can often be simplified to become more accessible or a student method can be used as is or adapted by the teacher or other students to become mathematically-desirable and accessible.

A final source of coherence and balance in the model is that it is helpful for subject areas other than math. Our CMW teachers often reported that Math Talk spread to an increased focus on inquiry and explanations in other subject areas. Students also spontaneously gave responsive assistance for other subject areas.

ENGLISH LANGUAGE LEARNERS

The Class Learning Path Model is effective with students from all backgrounds. But it especially simplifies the teacher's complex tasks in teaching students who must learn English as they are learning math. Garrison, Amaral, and Ponce (2006) describe their adaptation and use with teachers of Cummins' (1994) four quadrants in the LATCH model. These four quadrants are created by two axes (concrete to abstract solution strategies and context embedded to context reduced language) that result in four kinds of individual instruction teachers need to deliver within the classroom. The Class Learning Path Model simplifies this approach because all students are reached simultaneously and contribute to each other's learning. Learning for everyone initially has concrete solution strategies (e.g., Math Drawings linked to methods using formal math notation of some kind) and context embedded problems. Students higher in math skills will introduce more-advanced methods into the classroom discourse, and students higher in English will provide more-advanced explanations (that still may need to be extended by the teacher for full explanations). Thus, the English skills are modeled by classmates, and all students then need opportunities in classes to produce the relevant English words as rapid oral drills or other whole-class activities or in explanations of a solution method. As students gain experience in the topic, the problems become context reduced so as to generalize the math topic concepts. Therefore, the nice activities in the LATCH workshop outlined in the paper that have teachers sharing strategies that go into each quadrant now can go within the phases of the model: ways to create embedded context and concrete solution strategies go in Phase 1 and ways to reduce the context will be used in Phase 3 (see the LATCH Figure 2 for examples).

The effectiveness of the Class Learning Path Model in increasing English performance about math topics was indicated recently when students in a CMW school with many students identified as needing bilingual support were interviewed using the state interview of English speaking in academic areas and in everyday language. The interviewers were struck by the high levels of English students used to explain math concepts when they did not even know English words for parts of the body and other everyday English language. The teachers explained that in their Math Talk classrooms all students were expected to be able to learn to explain their thinking in English, and that with considerable modeling, they learned to do so.

CONCLUSION

Vygotsky's (1978) concept of a zone of proximal development for each student for each topic seems overwhelming to a teacher with as many as 35 students in a class (or even with "only" 20 different individuals). It suggests the need for total individualization and few whole-class activities. However, our research experience in many different classrooms for many different math topics over many years led to our simplified concept of a Class Zone of Proximal Development that operates within the four Class Learning Zone Teaching Phases to meet the needs of most students in the class by whole-class activities supported by the seven Responsive Means of Assistance within the emotional and cognitive supports of the Nurturing Meaning-Making Math Talk Community. The actual number of ways of thinking about a given situation are limited, so most or all can be discussed and examined as a way to understand the topic more deeply. The Inquiry Learning Path Teaching ensures that students are moving forward in their own learning path toward a mathematically-desirable and accessible general method. Those students who begin by knowing such a method increase their knowledge by explaining how multiple methods relate to each other and by assisting other students and the teacher within the interdependent Class Learning Zone created by the common learning supports within the Nurturing Meaning-Making Math Talk Community. Educational leaders can use the Class Learning Zone Model, with its integration of the principles from two NRC reports and from the NCTM Process Standards, to help teachers individualize their instruction to meet needs of their students within whole-class instruction.

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