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Standards for Computational Fluency: A Comparison of State and CCSS-M Expectations

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The adoption of *Common Core State Standards for Mathematics* (CCSS-M) by nearly all of the states (except Alaska, Minnesota, Texas, Virginia) represents an historic landmark in curriculum governance in the U.S. In just over two decades the U.S. has moved from a vision and standards for school mathematics developed by the National Council of Teachers of Mathematics (*Curriculum and Evaluation Standards for School Mathematics, 1989*) to consensus by nearly all of the states on grade-level mathematics learning goals for K-8 students.

While consensus on mathematics standards alone will not improve learning opportunities for students or provide necessary support for teachers, the standards do represent an important element in a standards-based reform system for nationwide improvement (Goertz, 2010). The CCSS-M aligned assessments under development by two state consortia (PARCC and SMARTER Balanced Assessment) have the potential to provide the other "bookend" of the standards system (Confrey & Krupa, 2010). What lies between these bookends (mathematics standards and aligned accountability assessments) is the important work of teachers, curriculum developers, and instructional leaders. In this regard CCSS-M provides an important opportunity for mathematics educators.

In 2006 a team from the Center for the Study of Mathematics Curriculum (CSMC) conducted a content analysis of a set of topics within the K-8 sections of 42 state standards documents. While not a comprehensive analysis of all the standards, a few topics were selected for in-depth analysis. One of these topics was computational fluency. Specifically, the team investigated when, according to the state standards, students are expected to begin study of computational methods and over what period of time (grades) they are expected to acquire fluency with whole number and fraction computation.

The study verified what many had believed—that state standards varied considerably with regard to the grade placement and language used to describe key learning goals K-8. The lack of consensus across the state standards is conjectured to be one of the primary causes for the repetitive nature of many of the textbooks available to schools and teachers over the past two decades. A full report of the CSMC analysis is available elsewhere (Reys, 2006).

In this article we compare learning expectations included in state standards, as described in 2006, regarding computational fluency with those outlined in CCSS-M. This summary is limited to four specific computational goals of school mathematics. Specifically, we summarize how the state and CCSS-M standards compare with regard to fluency with:

- Basic Number Combinations (Basic Facts)
- Multi-digit Whole Number Computation
- Fraction Computation
- Decimal Computation

In each case, we begin with findings from the earlier analysis of state standards. We then compare these findings to the set of parallel expectations (standards) articulated in CCSS-M.

Documentation of the changes in grade placement or nature of expectation will highlight areas where specific work is needed in both curriculum development and instructional planning.

Fluency with Basic Number Combinations (Basic Facts)

Basic number combinations are generally defined as the set of single-digit addition (or multiplication) combinations and their related subtraction (or division) combinations (e.g., 4+5=9; 9-4=5; 3x8=24; 24÷3=8). Fluency is the goal. That is, students are expected to recall quickly the sum, difference, product or quotient so that it can be used in various contexts, including problem solving, without undue delay.

Addition and Subtraction - Basic Number Combinations. In 2006, state standards articulated expectations regarding fluency in a variety of ways and sometimes at different grade levels. For example,

- Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory. (California, Grade 1)
- Demonstrate computational fluency for basic addition and subtraction facts with sums through 18 and differences with minuends through 18, using horizontal and vertical forms (Alabama Department of Education, 2003, Grade 2)
- State addition and subtraction facts. (Arizona Department of Education, 2003, Grade 2)

In addition to using particular language (e.g., state; commit to memory; fluency, horizontal and vertical forms) to specify the nature and parameters of the expectation, state standards also included guidance regarding when the work should start and often suggested initial parameters (number size or strategies) of development. For example, the Arkansas Department of Education Standards (2004) included the following progression spanning Kindergarten to Grade 2:

Kindergarten:

- Develop strategies for basic addition facts (counting all; counting on; one more, two more)
- Develop strategies for basic subtraction facts (counting back; one less, two less)

Grade 1:

- Develop strategies for basic addition facts (counting all; counting on; one more, two more; doubles; doubles plus one or minus one; make ten; using ten frames; Identity Property (add zero)
- Develop strategies for basic subtraction facts (relating to addition Ex. Think of 7 – 3 =____as "3 +___= 7"; one less, two less; all but one Ex.9 – 8, 6 – 5; using ten frames of the answers

Grade 2:

- Develop strategies for basic addition facts (counting all; counting on; one more, two more; doubles; doubles plus one or minus one; make ten; using ten frames; Identity Property (add zero) [standard repeated from Grade 1]
- Demonstrate computational fluency (accuracy, efficiency and flexibility) in addition facts with addends through 9 and corresponding subtractions (Ex. 9+9=18 and 18-9=9 add and subtract multiples of ten)

The 2006 analysis revealed that in most states, attention to developing fluency with basic number combinations began one or more years prior to when proficiency/fluency was expected. For example, 25 states introduced addition of basic number combinations in first grade. Introduction generally referred to exploration of strategies and/or focusing on a subset of number combinations. In 8 states fluency was expected by the end of grade 1. More states (27) specified this expectation by the end of Grade 2 and two states denoted it for Grade 3 (see Reys, 2006 for a detailed summary).

In CCSS-M the set of expectations and their progression regarding addition/subtraction of basic number combinations are as follows:

Kindergarten:

- Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem. *(K.OA.2[1])*
- Fluently add and subtract within 5. *(K.OA.2)*

Grade 1:

- Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing,

[1] The labeling of standards used in this article is as they appear in the CCSS-M document.

with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem. (1.OA.1)

- Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13). *(1.OA.6)*
- Develop strategies for basic subtraction facts (relating to addition Ex. Think of 7 – 3 =____as "3 +___= 7"; one less, two less; all but one Ex.9 – 8, 6 – 5; using ten frames of the answers

Grade 2:

- Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers. *(2.OA.2)*

While the topic is generally introduced in Grade 1 in the state standards, it begins in Kindergarten in CCSS-M. The CCSS-M standards, like many state standards, clearly emphasize the importance of acquiring fluency by building onto and using physical and mental models and strategies based on conceptual understanding. The CCSS-M grade specification for fluency with addition and subtraction basic number combinations (Grade 2) is consistent with the expectation and grade placement of a majority of state standards in 2006. However, the terminology used within CCSS-M to denote particular number parameters (e.g., "add and subtract **within 10**" (emphasis added)) is not common in state standards. This expression is defined in the glossary of CCSS-M (p. 85) as:

> ***Addition and subtraction* within *5, 10, 20, 100, or 1000.*** Addition or subtraction of two whole numbers with whole number answers, and with sum or minuend in the range 0-5, 0-10, 0-20, or 0-100, respectively. Example: 8 + 2 = 10 is an addition within 10, 14 – 5 = 9 is a subtraction within 20, and 55 – 18 = 37 is a subtraction within 100.

Multiplication and Division - Basic Number Combinations. In 2006 most state standards denoted fluency with basic facts for multiplication in Grade 3 (13 states) or Grade 4 (22 states) and for division in Grade 4 (20 states). As with addition and subtraction, state standards specify introduction of this topic, generally with a subset of number combinations, one or more grades prior to the expectation of fluency.

In CCSS-M expectations related to fluency with multiplication and division combinations are concentrated in Grade 3. Two standards convey the expectation. The first emphasizes use of strategies based on applications of arithmetic properties:

- Apply properties of operations as strategies to multiply and divide. Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.) *(3.OA.5)*

The second standard conveys the expectation of fluency (know from memory) by the end of Grade 3:

- Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers. *(3.OA.7)*

A comparison of the CCSS-M progression with state standards reveals a shift for most states. That is, in 2006 24 states expected students to acquire fluency with multiplication and division basic number combinations at a later grade, generally at Grade 4. Likewise, in the majority of state standards in 2006, acquisition of fluency with multiplication and division combinations is spread over at least two years (Grades 2 and 3 or Grades 3 and 4) while in CCSS-M this work is concentrated in Grade 3, with no mention in prior grades.[2]

[2] There is preliminary work on relating addition and multiplication as operations in Grade 2.

Table 1

GRADE PLACEMENT OF LEARNING EXPECTATIONS RELATED TO FLUENCY WITH BASIC NUMBER COMBINATIONS FOR EACH OPERATION[3]					
Operation	**Grade**	**Number of States (N=39)**	**Operation**	**Grade**	**Number of States (N=39)**
Addition	1	8	Subtraction	1	7
	2	28 (CCSS-M)		2	27 (CCSS-M)
	3	2		3	3
	Not specified	1		Not specified	2
Multiplication	3	13 (CCSS-M)	Division	3	6 (CCSS-M)
	4	22		4	20
	5	1		5	3
	6	1		6	1
	Not specified	2		Not specified	9

Table 1 summarizes the grade placement of the expectation for fluency with basic number combinations for each operation across states in 2006 as well as within CCSS-M. As noted, the grade at which fluency with addition and subtraction number combinations was expected in 2006 is generally consistent with CCSS-M (Grade 2). However, CCSS-M specifies fluency with multiplication and division number combinations about one year earlier than most state standards and concentrates this work within one grade level (Grade 3).

Multi-Digit Whole Number Computation

Addition and Subtraction. Across the set of 42 state standards reviewed in 2006, the progression leading to fluency with multi-digit addition varied considerably. For example, in some states, students were expected to begin adding multi-digit numbers as early as Kindergarten or as late as Grade 3. The culminating standard (grade at which fluency was expected) ranged from Grade 1 to 6. The span within a given state between when the topic was introduced and when fluency was expected ranged from 1 to 4 grades, with two or three grades the most typical span.

In CCSS-M the progression towards fluency with multi-digit addition of whole numbers begins in Grade 1 with the following standard:

- Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten. *(CCSS-M 1.NBT.4)*

The culminating standard for multi-digit whole number addition is in Grade 4:

- Fluently add and subtract multi-digit whole numbers using the standard algorithm. *(CCSS-M 4.NBT.4)*

The complete set of CCSS-M standards related to multi-digit whole number addition is shown in Table 2 (see next page).

Multiplication and Division. For multiplication of multi-digit whole numbers CCSS-M specifies a progression of standards (see Table 3, page 26), starting in Grade 3. Specifically, students are expected to first use strategies

[3] Thirty-nine state standards were included in the analysis of fluency with basic number combinations (three other states in the main analysis did not include standards for primary grades).

based on place value and arithmetic properties to multiply 1-digit whole numbers by multiples of 10 (less than 100). Strategy use continues in Grade 4 with more complex combinations (multiply 1-digit whole numbers by up to 4-digit whole numbers and 2-digit whole numbers by 2-digit whole numbers). The culminating standard is noted in Grade 5, specifying "fluency multiplying multi-digit whole numbers using the standard algorithm."

This progression is similar to what a few states outlined in 2006. For example, in most states, multiplication of multi-digit whole numbers began in Grade 3 (19 states) or

Table 2

PROGRESSION OF STANDARDS RELATED TO MULTI-DIGIT WHOLE NUMBER ADDITION/SUBTRACTION IN CCSS-M	
Grade 1	Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten. *(1.NBT.4)*
	Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used. *(1.NBT.5)*
	Subtract multiples of 10 in the range 10-90 from multiples of 10 in the range 10-90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. *(1.NBT.6)*
Grade 2	Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. *(2.OA.1)*
	Fluently add and subtract within 20 using mental strategies. *(2.OA.2)*
	Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. *(2.NBT.5)*
	Add up to four two-digit numbers using strategies based on place value and properties of operations. (2.NBT.6)
	Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three- digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds. *(2.NBT.7)*
	Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900. (2.NBT.8)
	Explain why addition and subtraction strategies work, using place value and the properties of operations. *(2.NBT.9)*
	Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g. by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem. *(2.MD.2)*
Grade 3	Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. *(3.NBT.2)*
Grade 4	Fluently add and subtract multi-digit whole numbers using the standard algorithm. *(4.NBT.4)*

Table 3

PROGRESSION OF STANDARDS RELATED TO MULTI-DIGIT WHOLE NUMBER MULTIPLICATION/DIVISION IN CCSS-M	
Grade 3	Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations. (3.NBT.3)
Grade 4	Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. (4.NBT.5)
Grade 5	Fluently multiply multi-digit whole numbers using the standard algorithm. (5.NBT.5)

Grade 4 (21 states). In about half the state standards, fluency is expected in Grade 4 (20 states). In 15 states, fluency is expected in Grade 5.

In general, the progression outlined in CCSS-M for multi-digit whole number multiplication is similar to that outlined in most state documents, although the work is generally condensed into two rather than 3 years. One difference between CCSS-M and most state standards is the reference to "the standard algorithm." The glossary of CCSS-M defines "computational algorithm" as, "A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly." However, no definition or description for "the standard algorithm" is provided.

The first CCSS-M standard regarding whole number division (multi-digit numbers) is at Grade 4:

- Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. *(4.NBT.6)*

In Grade 5 the expectation is extended:

- Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. *(5.NBT.6)*

The culminating expectation is found at Grade 6, again by demonstration of fluency with "the standard algorithm:"

- Fluently divide multi-digit numbers using the standard algorithm. *(6.NS.2)*

Table 4 (see next page) includes a summary of the grade at which the culminating standard (expectation for fluency) for each operation (addition, subtraction, multiplication and division) is denoted across state standards, based on the 2006 analysis. Also noted is the grade within CCSS-M where the expectation of fluency is noted. *In general, the grade at which CCSS-M specifies multi-digit whole number fluency with each operation is at or slightly later than specified in 2006 in most state standards documents.*

As noted earlier, the specific statement of the culminating standard for each operation in CCSS-M includes the expectation of use of "the standard algorithm." For example:

- Fluently add and subtract multi-digit whole numbers using the standard algorithm. *(4.NBT.4)*
- Fluently multiply multi-digit whole numbers using the standard algorithm. *(5.NBT.5)*
- Fluently divide multi-digit numbers using the standard algorithm. *(6.NS.2)*

However, a definition for "the standard algorithm" is not offered. If the authors of CCSS-M had a particular standard algorithm in mind, it was not made explicit nor is an argument offered for why a particular (standard) algorithm is expected.

Fraction Computation

Tables 5 and 6 (see pages 28-29) include all of the CCSS-M standards related to computation with fractions. As noted in Table 5, the CCSS-M standards related to addition and subtraction of fractions begin with addition of like denominators (Grade 4) followed by denominators of 10 and 100 (Grade 4) then focus on the more general

Table 4

GRADE PLACEMENT FOR CULMINATING LEARNING EXPECTATIONS RELATED TO FLUENCY WITH WHOLE NUMBER COMPUTATION FOR EACH OPERATION (N=42)					
Operation	**Grade**	**Number of States**	**Operation**	**Grade**	**Number of States**
Addition	1	8	Subtraction	1	1
	2	3		2	2
	3	14		3	15
	4	15 (CCSS-M)		4	15 (CCSS-M)
	5	5		5	5
	6	3		6	3
	N/S*	11		N/S	1
Multiplication	3	2	Division	3	0
	4	21		4	12
	5	15 (CCSS-M)		5	23
	6	3		6	6 (CCSS-M)
	N/S	1		N/S	1

**Not specified within state document.*

case, unlike denominators, by use of equivalent fractions in Grade 5. Unlike computation with whole numbers, CCSS-M standards related to addition and subtraction of fractions do not use the term "standard algorithm." However, a specific strategy, including its general form, is included in the statement of the standard:

> *Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)* (5.NF.1)

A general finding in the analysis of state standards in 2006 was variability in the grade levels at which addition, subtraction, multiplication, and division of fractions was introduced (see Table 7, pg. 30) and when fluency was expected (see Table 8, pg. 30). As noted in Table 7, more than half of the states' standards reviewed introduced addition and subtraction of fractions in Grade 4 and this is also where the first standard in CCSS-M related to this topic is found:

> *Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.* (CCSS-M, 4.NF.3c)

However, CCSS-M deviates from all but one state by introducing multiplication of fractions in Grade 4. As Table 7 indicates, the majority of states (25 of 42) introduce multiplication of fractions two years later, in Grade 6. In fact, it is customary in most state standards to develop both multiplication and division of fractions in the same grade. CCSS-M's introduction of fraction division in Grade 5 represents acceleration of grade placement in all but 7 states.

In general, the timeline from introduction of the concept of fraction to beginning computation with fractions, then to expectation of fraction computation fluency is more condensed in CCSS-M than in state standards reviewed in 2006. For example, the first standard related to the concept of fraction in CCSS-M is at Grade 3; addition, subtraction, and multiplication of fractions is introduced in Grade 4;

and fluency is expected for these three operations on fractions in Grade 5. Introduction of fraction division appears in Grade 5 in CCSS-M, followed by expectation for fluency the following year, in Grade 6.

Decimal Computation

Relatively few CCSS-M standards address decimal computation; in fact, only two standards (one in Grade 5 and the other in Grade 6) specifically identify computation with decimals:

> *Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.* (CCSS-M 5.NBT.7)

> *Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.* (CCSS-M 6.NS.3)

Subsequent standards in Grade 7 use the more general set of "rational numbers" when specifying computation expectations. For example:

> *Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.* (7.EE.3)

As noted, the second standard (CCSS-M 6.NS.3) refers to "the standard algorithm" without explicitly defining this strategy. CCSS-M also calls for work with fraction computation to precede that of decimal computation. Some state standards in 2006 mirrored this approach while others reversed the order, emphasizing computation with decimals prior to that of fractions, likely in order to build upon the similar strategies of whole number computation.

Table 5

PROGRESSION OF STANDARDS RELATED TO FRACTIONAL ADDITION AND SUBTRACTION IN CCSS-M	
Grade 4	Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction. *(4.NF.3c)*
	Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem. *(4.NF.3d)*
	Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. For example, express 3/10 as 30/100, and add 3/10 + 4/100 = 34/100. *(4.NF.5)*
Grade 5	Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.) *(5.NF.1)*
	Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 < 1/2. *(5.NF.2)*
Grade 7	Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram. *(7.NS.1)*
	Solve real-world and mathematical problems involving the four operations with rational numbers. *(7.NS.3)*

Table 6

PROGRESSION OF STANDARDS RELATED TO FRACTIONAL MULTIPLICATION AND DIVISION IN CCSS-M.	
Grade 4	Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie? *(4.NF.4c)*
Grade 5	Interpret a fraction as division of the numerator by the denominator (a/b = a ÷ b). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie? *(5.NF.3)*
	Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction. *(5.NF.4)*
	Interpret multiplication as scaling (resizing) by: Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence a/b = (nxa)/(nxb) to the effect of multiplying a/b by 1. *(5.NF.5b)*
	Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *(5.NF.6)*
	Interpret division of a unit fraction by a non-zero whole number and compute such quotients. For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3. *(5.NF.7a)*
	Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4. *(5.NF.7b)*
	Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. For example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins? *(5.NF.7c)*
Grade 6	Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi? *(6.NS.1)*
Grade 7	Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers. *(7.NS.2)*
	Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as (-1)(-1) = 1 and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts. *(7.NS.2a)*
	Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then –(p/q) = (-p)/q = p/(-q). Interpret quotients of rational numbers by describing real-world contexts. *(7.NS.2b)*
	Apply properties of operations as strategies to multiply and divide rational numbers. *(7.NS.2c)*
	Solve real-world and mathematical problems involving the four operations with rational numbers. *(7.NS.3)*

Table 7

NUMBER AND GRADE-LEVEL WHEN STATE STANDARDS INTRODUCE EXPECTATIONS RELATED TO COMPUTATION WITH FRACTIONS			
Grade	**Addition & Subtraction of Fractions**	**Multiplication of Fractions**	**Division of Fractions**
1st grade	2 states		
2nd grade			
3rd grade	7 states		
4th grade	22 states (CCSS-M)	1 state (CCSS-M)	1 state
5th grade	9 states	10 states	6 states (CCSS-M)
6th grade	1 state	25 states	27 states
7th grade	1 state	5 states	6 states

Table 8

NUMBER OF STATES AND GRADE-LEVEL WHEN STATE STANDARDS INDICATE EXPECTATION OF FLUENCY WITH ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION OF FRACTIONS[4]			
Grade	**Addition & Subtraction of Fractions**	**Multiplication of Fractions**	**Division of Fractions**
4th grade	1 state		
5th grade	15 states (CCSS-M)	2 states (CCSS-M)	1 state
6th grade	20 states	25 states	24 states (CCSS-M)
7th grade	6 states	13 states	14 states
8th grade		1 state	1 state

According to the 2006 state standards, computational fluency with decimals was to be developed in the late elementary and early middle school years, spanning from Grade 4 to Grade 7 depending upon the particular state. In CCSS-M, fluency with decimal operations is expected one year after students are first introduced to decimals. This represents another potentially condensed progression toward computational fluency.

Summary

The purpose of this discussion is to highlight within one area of the K-8 mathematics curriculum where CCSS-M represents a shift in current practice. The general findings of this analysis include:

- CCSS-M specifies computational fluency in the following areas in the SAME grade level as most state standards in use in 2006:
  - Addition and subtraction of basic number facts (Grade 2);
  - Addition and subtraction of multi-digit whole numbers (Grade 4);
  - Division of fractions (Grade 6).
- CCSS-M specifies computational fluency in the following areas at an EARLIER grade level compared to most state standards in use in 2006:
  - Multiplication and division of basic number facts (Grade 3);

[4] For this analysis, we used the culminating learning expectation that indicated students were working with common and uncommon denominators when adding and subtracting fractions.

Table 9

GRADE LEVEL AT WHICH CCSS-M INDICATES EXPECTATION OF COMPUTATIONAL FLUENCY						
	Gr1	**Gr2**	**Gr3**	**Gr4**	**Gr5**	**Gr6**
Basic Facts – Addition/Subtraction		x				
Basic Facts – Multiplication/Division			x			
Multi-digit numbers –Addition/Subtraction				x		
Multi-digit numbers – Multiplication					x	
Multi-digit numbers – Division						x
Fractions – Addition/Subtraction					x	
Fractions – Multiplication					x	
Fractions – Division						x
Decimals – Addition/Subtraction						x
Decimals – Multiplication/Division						x

  – Addition, subtraction and multiplication of fractions (Grade 5).

- CCSS-M specifies computational fluency in the following areas at a LATER grade level compared to most state standards in use in 2006:
  – Multiplication of multi-digit numbers (Grade 5);
  – Division of multi-digit numbers (Grade 6).
- Attention to fluency with basic facts, whole number multiplication, and fractional operations is condensed over a shorter grade span in CCSS-M than in the majority of state standards in use in 2006.
- Decimal computation receives very little attention in CCSS-M (two standards) and follows the development of fraction computation.
- In several instances, CCSS-M specifies use of a 'standard' algorithm (e.g. for whole number computation) without specification of meaning of the specific algorithm intended. For fraction computation, a particular strategy is specified and associated with fluency.
- There is considerable attention in CCSS to applying computational procedures to "word problems."
- Prior to an expectation of fluency, CCSS-M standards specify use of models and individually derived strategies for computing.

This analysis confirms that there is a concentration of work on computation of whole numbers, fractions and decimals in Grades 3-6 (see Table 9). The majority of work with fractions is expected one year earlier than in most state standards published in 2006. Part of the rational for this shift is to "make room for" new emphasis on particular mathematics content (e.g., algebra) introduced and developed in the middle grades. However, given that this is a new roadmap for school mathematics and that there will be a significant transition period, it is not clear when or if middle grade teachers can count on students entering middle grades ready to undertake the advanced content.

As noted, this discussion has focused exclusively on computation. It is likely that in other topic areas there are similar or perhaps more dramatic shifts in learning goals at particular grades. These shifts will need to be carefully outlined and discussed by teachers and they will need new curriculum materials to help guide instruction designed to support student learning of the goals of CCSS-M.

CCSS-M provides a rather ambitious roadmap for changes in the K-8 school mathematics program - an important ingredient for systemic and widespread mathematics reform efforts. A strong formative and summative assessment system aligned with CCSS-M is also needed and under development by two state-led consortia. The next steps include formulating a system for closely monitoring and adjusting CCSS-M so that common standards represent a continually improving roadmap for intended learning goals. Equally important are curriculum resource development and large-scale efforts to improve teaching. As a leading educator has recently argued, unless there is a concerted effort to make improvements in teaching of mathematics, it is difficult to see how the goals of CCSS-M will be realized. (Ball, 2011).

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