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Call for Manuscripts

The editors of the ***NCSM Journal of Mathematics Education Leadership*** are interested in manuscripts addressing issues of leadership in mathematics education and reflecting a broad spectrum of formal and informal leadership at all levels. Categories for submittal include:

- **Key topics** in leadership and leadership development
- **Case studies** of mathematics education leadership work in schools and districts or at the state level and the lessons learned from this work
- **Reflections** on what it means to be a mathematics education leader and what it means to strengthen one's leadership practice
- **Research reports** with implications for mathematics education leaders
- **Professional development efforts** including how these efforts are situated in the larger context of professional development and implications for leadership practice
- **Brief commentaries on critical issues** in mathematics education
- **Brief reviews of books** that would be of interest to mathematics education leaders

Other categories that support the mission of the journal will also be considered. Currently, the editors are particularly interested in manuscripts that address the leadership work of mathematics coaches and mathematics specialists.

We also invite readers to submit letters to the editor regarding any of the articles published in the journal. We want to hear about your reactions, questions, and connections you are finding to your work. Selected letters will be published in the journal with your permission.

Submission/Review Procedures

Submittal of manuscripts should be done electronically to the ***Journal*** editor, currently Linda Ruiz Davenport, at ncsmJMEL@mathedleadership.org. Submission should include (1) one Word file with the body of the manuscript without any author identification and (2) a second Word file with author information as you would like it to appear in the journal. Each manuscript will be reviewed by two volunteer reviewers and a member of the editorial panel.*

NATIONAL COUNCIL OF SUPERVISORS OF MATHEMATICS (NCSM)

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***Note:** Information for manuscript reviewers can be found on the inside back cover of this publication.

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Purpose Statement

The ***NCSM Journal of Mathematics Education Leadership*** is published at least twice yearly, in the spring and fall. Its purpose is to advance the mission and vision of the National Council of Supervisors of Mathematics by:

- Strengthening mathematics education leadership through the dissemination of knowledge related to research, issues, trends, programs, policy, and practice in mathematics education
- Fostering inquiry into key challenges of mathematics education leadership
- Raising awareness about key challenges of mathematics education leadership, in order to influence research, programs, policy, and practice
- Engaging the attention and support of other education stakeholders, and business and government, in order to broaden as well as strengthen mathematics education leadership.

Comments from the Editors

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Some time has now passed since the CCSS for Mathematics were finalized and adopted by many of our states. Where the CCSS for Mathematics has been adopted, states, districts, and schools are working hard to consider what these new standards mean for mathematics teaching and learning in their classrooms. What curriculum resources might be needed to fully address these standards? What professional development might be helpful to teachers and administrators? What about the coming PARCC or Smarter Balanced assessments and what might be the implications for the kinds of ongoing assessments to monitor student progress? Are we sure we even understand the standards themselves? There are indeed a myriad of questions still to be answered. Even in states where the CCSS for Mathematics were not adopted, many are examining these new standards to learn how they might serve to strengthen mathematics teaching and learning at a district or school level. Their focus, coherence, and rigor may have a great deal to offer regardless of whether they are formally adopted at the state level. In this issue of the Journal for Mathematics Education Leaders, we hear from mathematics education leaders about questions that are arising during this process of transition and what some beginning efforts to support teachers during this transition can offer us.

What happened to the CCSS for Mathematics as these standards were adopted by states? In this article, Barbara Reys and her team of authors take a look at how some states augmented or annotated the CCSS for Mathematics in order to give them a strong state identity, how they collaborated with assessment consortia to design and utilize common assessments, and how they developed and began to implement a timeline for transition from earlier state frameworks to their new frameworks. The authors raise important questions about the extent to which these activities at the state level are making their way into districts, how these activities get played out in schools, and what the implications might be for mathematics education leaders at all these different levels.

One important question for the transition to the CCSS for Mathematics at the state, district, and school level is what is meant by "the standard algorithm." Karen Fuson and Sybilla Beckman take on this question by examining the standards that reference standard algorithms, how these are discussed and described in the Number and Base Ten Progression document, what the authors of the CCSS for Mathematics have said about standard algorithms, and how this lines up with what we know about the number of variations of algorithms that are used across the country and internationally. Their analysis of what is meant by "algorithm" opens the door to an important discussion of how much variation in a written method for a standard algorithm might be reasonable. They also point out, importantly, that discussions of these variations in standard algorithms, as well as the general methods that underlie them, are important to explore and discuss with students in order to support understanding and contributes to the development of fluency.

We have the opportunity to see how teachers and administrators participating in the Greater Birmingham Mathematics Partnership (GBMP) are transitioning to the CCSS for Mathematics. Here, an author team from Birmingham-Southern College, the Mathematics Education Collaborative, Montgomery College, and the University of Alabama at Birmingham discuss how they use the notion of *Challenging Courses and Curricula* to shape a K-12 professional development model to promote instruction consistent with the CCSS for Mathematics. They define "challenging courses" to be those in which students develop expertise with the Standards for Mathematical Practice, and in their article, they provide examples of the kind of classroom practice that leads to this expertise. Their examples address "big mathematical ideas," inquiry and reflection, productive disposition, and communication. This article helps us think about how more teachers can be supported in taking on this vision of classroom practice and the important role of mathematics education leaders in providing that support.

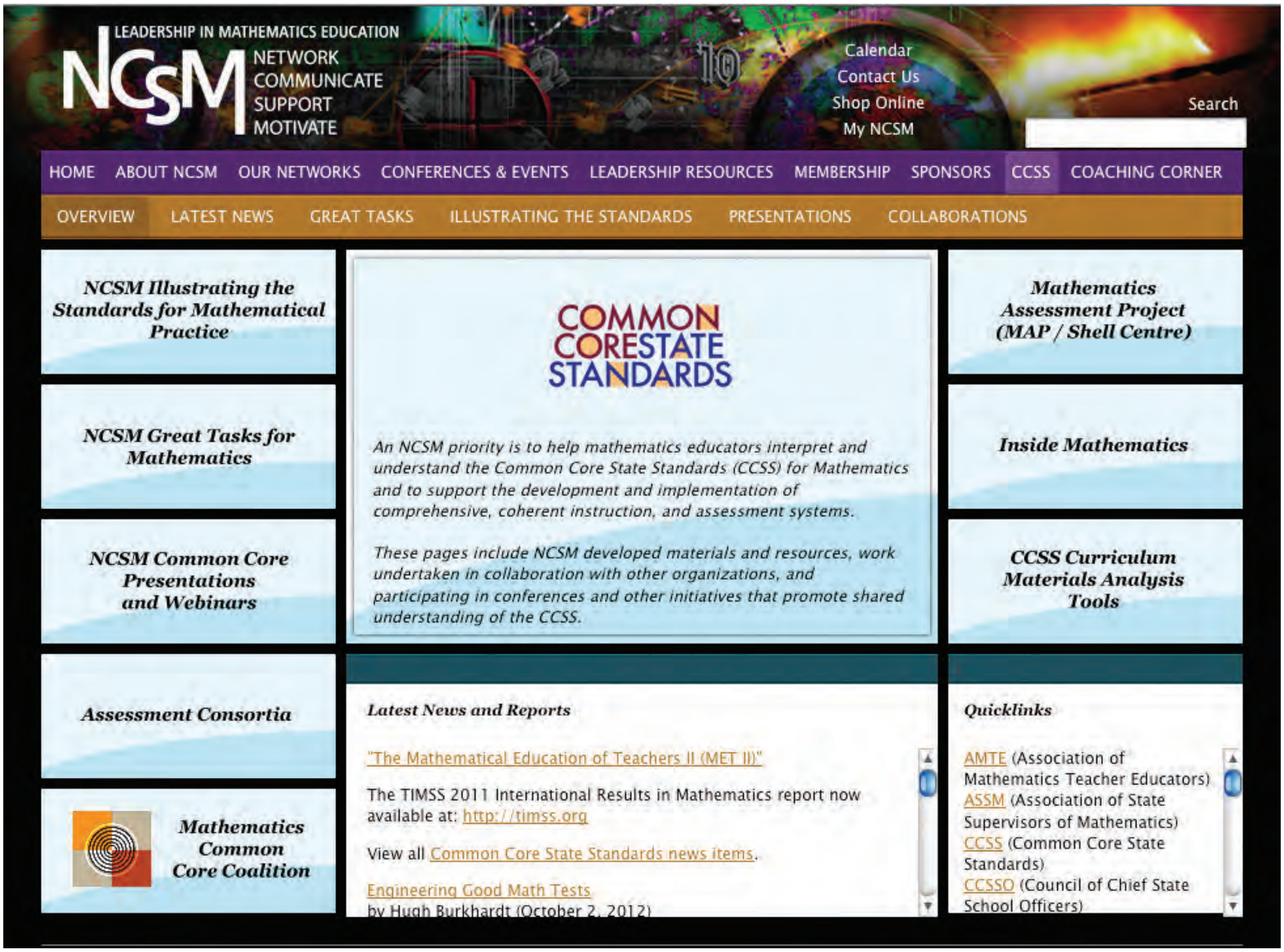
The role of context in solving fraction problems, given Mathematical Practice Standard #2: *Reason abstractly and quantitatively*, is examined in an article by Travis and Melfried Olson. They focus on how context can be used to support the formulation of models and representations that can then shape solution strategies to what they call the Painting Problem: *It takes 3/4 liter of paint to cover 3/5 m^2. How much paint is needed to paint 1 m^2?* They examine how selected students and teachers approached this problem and make an argument that the use of models and representations in certain contexts contributed to the conceptual development of key algorithms. In their conclusion they discuss how mathematics education leaders can help teachers and students learn to use models and representations to reason through problems like the Paint Problem.

In our final article from a team from the Rice University School Mathematics Project, we learn about the role of master teachers in its Summer Campus Program, where selected K-12 classroom teacher leaders serve as instructors, role models, and mentors for teachers participating in the project as they help these teachers develop the essential understandings embedded in the CCSS for Mathematics. Data on the impact of the program suggest a positive impact on participants' self-efficacy and preparedness to teach mathematics. The article offers insights into how effective teacher professional development designed to support the transition to the CCSS for mathematics might be designed.

Fortunately, many resources are being created to help schools, districts, and states transition to the CCSS for Mathematics. At the NCSM web page (http://www.mathedleadership.org/ccss/index.html) the CCSS link provides materials and resources developed by NCSM and other organizations working in collaboration with NCSM designed to help promote a shared understanding of the these new standards and their implications. This includes the NCSM *Illustrating the Standards for Mathematical Practice*, the NCSM *Great Tasks for Mathematics*, and a number of NCSM Common Core presentations and webinars. From this page you will also see links to the assessment consortia, the *Mathematics Assessment Project* (MAP) from the University of California at Berkeley and the Shell Centre team at the University of Nottingham, and *Inside Mathematics*, a professional development resource that includes tools for mathematics instruction and classroom examples of mathematics instruction that reflects the expectations of the CCSS for Mathematics. There is also a link to a materials analysis tool, initiated at the request of the Council of Chief State School Officers (CCSSO) and led by Bill Bush at the University of Louisville, designed to assess the potential of curriculum materials to support student attainment of the CCSS for Mathematics.

Other organizations are also developing and hosting similar links to resources on their web pages. For instance, at the Council for Great City Schools (CGCS) web page (http://www.cgcs.org/domain/94) we see a link to a variety of resources including an introduction to the CCSS for Mathematics, a discussion of the instructional shifts associated with the standards, professional development videos that include a discussion of the standards, links to the Math Progression documents, links to online professional development modules, a brief overview of the assessment consortia with links to their web pages, and *Parent Roadmaps* that lay out the expectations of the CCSS for parents. Some of this material is also available in Spanish. In addition, their links to other resources includes a many useful resources including a link to Mathematics Publisher's Criteria K-8 created to support implementation of the CCSS for Mathematics by providing specific criteria for aligned materials based on the design principles of focus and coherence. The CGCS home page also includes a report on how urban school districts are transitioning to the CCSS.

Projects like Illustrative Mathematics (http://www. illustrativemathematics.org) also contain helpful resources that "illustrate the range and types of mathematical work that students experience in a faithful implementation of the Common Core State Standards." These include tasks that illustrate the expectations of the mathematics content standards as well as videoclips that illustrate the mathematical practice standards. Another project, Implementing the Mathematical Practice Standards



The CCCS link at the NCSM web page: http://www.mathedleadership.org/ccss/index.html

(http://mathpractices.edc.org), provides illustrations of each of these standards that include a mathematics task; a student dialogue based on that task; information about grade level, standards, and the context for the dialogue; teacher reflection questions; a mathematical overview; and optional student materials.

Resources developed by states that adopted the CCSS for Mathematics are also increasingly available as state departments of education find ways to communicate about the expectations of these standards and how these expectations might be addressed. Many now include access to sample model curriculum units, sets of assessment tasks, or other materials.

Despite these emerging materials and resources, many questions remain as we make our transition to the CCSS for Mathematics. If you have a story to tell about your efforts to support this transition as a mathematics education leader, please consider sharing what you are learning with the broader mathematics education leadership community. We would love to hear from you!

Of course we also realize there are many stories to tell about the broad range of efforts to support mathematics teaching and learning in classrooms, schools, districts, and at the university level. There are many challenges to consider as we think about the mathematics learning of students with special needs, students whose first language is not English, students who enter schools in the United States with limited prior schooling, and students who are struggling with some aspects of the mathematics content they are expected to learn. There are challenges as we think about the needs of urban and rural settings and the diversity that can often be found in these settings. There is also much to think about as we consider the role of technology, and how these tools play a role in strengthening mathematics teaching and learning as well as how these tools might be used to strengthen our professional development with teachers, teacher leaders, and administrators. Please let us know about your work in these areas as well.

Finally, please consider attending the **NCSM Annual Conference in Denver, April 15-17**. This is an opportunity to enlarge your network of colleagues who share your interests and love of mathematics education, and hear their stories, whether in conference sessions or during informal conversations outside of sessions.

We hope you enjoy this issue of the *Journal of Mathematics Education Leadership*. We also hope it supports the very important mathematics education leadership work you are engaged in your respective sites. If you would like to comment on any of the articles, or raise a question for consideration by the authors, please send these on to us for the *Letters to the Editor* section of the journal. We hope that what we publish in all of our journal issues will serve to generate many rich and useful conversations.

State-Level Actions Following Adoption of Common Core State Standards for Mathematics

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The adoption of *Common Core State Standards for Mathematics* (CCSS-M) by 45 states,[1] the District of Columbia, the Northern Mariana Islands, and the U.S. Virgin Islands represents a historic landmark in curriculum governance in the United States. For the first time, a significant majority of K-12 teachers and students will focus on common learning expectations for mathematics. Coupled with common grade-level assessments aligned to CCSS-M currently under development by two state-led consortia—Partnership for Assessment of Readiness for Colleges and Careers (PARCC) and the Smarter Balance Assessment Consortium (SBAC)—this initiative has the potential to impact aspects of educational practice critical to K-12 students' mathematical learning (e.g., teacher preparation and professional development, curriculum material development, and policies related to K-12 course-taking and graduation requirements).

Adopting common mathematics standards was a significant undertaking and many are surprised at the widespread, rapid, and non-partisan acceptance of CCSS-M, particularly given the historic record of local (or state) governance with regard to educational decisions (Goertz, 2008; Long, 2003). The widespread acceptance of CCSS-M is due, at least in part, to the fact that the development of common standards was a state-driven initiative led by the Council of Chief State School Officers (CCSSO) and the National Governors Association (NGA).

Following the adoption of CCSS-M, state education systems have engaged in several initiatives, including:

- Giving a state identity to CCSS-M, in some cases augmenting CCSS-M according to local needs;
- Collaborating with one or both of the state-led assessment consortia to design and utilize common assessments for grades 3-8 and high school; and
- Developing and implementing a timeline and plan for transitioning from current state standards to CCSS-M.

This article provides a summary of state actions taken in the first year following adoption of CCSS-M in these three areas.

Giving CCSS-M a State Identity

Although states were expected to adopt CCSS-M in its entirety, thus, resulting in "common" standards across the U.S., they were granted latitude in order to honor local needs. As noted in information shared with states, "while states will not be considered to have adopted the common core if any individual standard is left out, states are allowed to augment the standards with an additional 15% of content that a state feels is imperative" (Achieve, 2010).

To date, 35 of the 45 states that adopted CCSS-M have done so without "augmenting" the standards. That is, they

[1] The CCSS was adopted by all states except Alaska, Minnesota, Nebraska, Texas, and Virginia.

adopted CCSS-M without adding additional standards or modifying the language of the standards. In these cases, the state departments of education websites either link directly to the standards located on the official CCSS-M website (http://www.corestandards.org/) or the states developed a new cover page/front material for the document with state identification (e.g., *Indiana Common Core State Standards* available at: https://learningconnection.doe.in.gov/Standards/About.aspx?art=11.

Ten states augmented CCSS-M prior to or immediately following its adoption. Eight of these states augmented CCSS-M by: (1) adding additional standards (Alabama, Arizona, California, Colorado, Iowa, Massachusetts, and

Table 1 — North Dakota Mathematics Content Standards

Kindergarten

Domain: Counting and Cardinality		**K.CC**
Cluster: Know number names and the count sequence.		
Code	**Standards**	**Annotation**
K.CC.1	Count to 100 by ones and by tens.	Pennies and dimes may be used to model ones and tens.
K.CC.2	Count forward beginning from a given number within the known sequence (instead of having to begin at 1).	Number range for this skill should be up to 100. Example: Student is given a number within the range of 0 to 100. For example, use 56. Student must count forward in sequence from that number. “56, 57, 58, 59” on so on.
K.CC.3	Write numbers from 0 to 20. Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects).	
Cluster: Count to tell the number of objects.		
Code	**Standards**	**Annotation**
K.CC.4	Understand the relationship between numbers and quantities; connect counting to cardinality.	Number range for this skill should be up to 20.
	a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.	
	b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.	
	c. Understand that each successive number name refers to a quantity that is one larger.	
K.CC.5	Count to answer “how many?” questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.	This standard includes the following skills: a. Use up to 20 objects arranged in a line, rectangular array and a circle. b. Use up to 10 objects in a scattered configuration. c. When given a number from 1-20, count out that many objects.

Source: (*North Dakota Mathematics Content Standards*, http://www.dpi.state.nd.us/standard/content/math/2011/math.pdf, p. 12)

New York) or encouraging districts to give more emphasis to specific topics (Kansas); (2) moving standards from one grade to another (California), or (3) modifying standards by adding or changing words (Alabama, California, Colorado). The other two states (Maryland and North Dakota) modified the format or annotated CCSS-M. In particular, North Dakota added an "annotations" column with examples, definitions, and comments in the state's CCSS-M document but did not change the individual statements of the standards within CCSS-M. The annotations are intended to help district administrators and teachers understand the standards and provide guidance in interpreting them. (See Table 1 for sample annotations from the *North Dakota Mathematics Content Standards,* Grade K.)

In the Maryland version of CCSS-M (Maryland Department of Education, 2011), statements of "Essential Skills and Knowledge" follow many of the common core standards. These statements are intended to:

> provide language to help teachers develop common understandings and valuable insights into what a student must know and be able to do to demonstrate proficiency with each standard. Maryland mathematics educators thoroughly reviewed the standards and, as needed, provided statements to help teachers comprehend the full intent of each standard. The wording of some standards is so clear, however, that only partial support or no additional support seems necessary. (p. 5)

For example, at Grade 3, following the standard (3.NF.1), *Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size 1/b*, Maryland (2011) includes the following additional statements:

- Knowledge of the relationship between the number of equal shares and the size of the share;
- Knowledge of equal shares of circles and rectangles divided into or partitioned into halves, thirds, and fourths;
- Knowledge that, for example, the fraction 1⁄4 is formed by 1 part of a whole which is divided into 4 equal parts. Knowledge that, for example, the fraction 3⁄4 is the same as 1⁄4 + 1⁄4 + 1⁄4 (3 parts of the whole when divided into fourths);
- Knowledge of the terms numerator (the number of parts being counted) and denominator (the total number of equal parts in the whole);
- Knowledge of and ability to explain and write fractions that represent one whole (e.g., 4/4, 3/3);
- Ability to identify and create fractions of a region and of a set, including the use of concrete materials; and
- Knowledge of the size or quantity of the original whole when working with fractional parts. (p. 18)

Table 2 includes a summary of the extent and nature of state augmentation of CCSS-M by eight states and includes examples of standards that were added, deleted, moved to a different grade level, or whose language was changed. As noted, in the Kansas version of CCSS-M teachers are encouraged to give additional attention to two themes: probability and statistics and algebraic patterning:

> In recognition of the long history in Kansas of the ability for local school districts to make decisions for themselves, the review committee felt strongly that these topics should be set aside from the detail of the main document with enough information provided for each school and/or district to decide how to incorporate [these topics]. (p. 9)

Common themes in the K-8 standards added (augmented) include:

- Emphasis on money or time in the primary grades (CA, IA, MA)
- Emphasis on computational estimation, judging reasonableness of computations, or approximate error in measurement (CA, MA)
- Increased attention to patterning (CA, KS)

Although five states (Alaska, Minnesota, Nebraska, Texas, and Virginia) have, to date, chosen not to adopt CCSS-M, the common core initiative is having an impact in at least some of these states. For example, the Virginia Department of Education (2011) website indicates that the state is:

> using the commonwealth's established process for adopting and revising academic standards to incorporate content from the *Common Core State Standards* into the Standards of Learning (SOL). In doing so, the board

Table 2. Examples of augmentation of CCSS-M

State	Extent and Nature of Augmentation	Examples of Augmentation
AL	65 changes (Gr. 9-12, including new standards and additional words added to CCSS-M standards)	• *Analyze determinants and inverses of 2 × 2, 3 × 3, and larger matrices to determine the nature of the solution set of the corresponding system of equations, including solving systems of equations in three variables by echelon row reduction and matrix inverse. (Gr. 9-12, new)* • *Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., v, |v|, ||v||),* ***including the use of eigen-values and eigen-vectors****. (Gr. 9-12, phrase in bold added)*
AZ	8 standards added (three at Gr. 4; one at Gr. 6; four at Gr. 9-12)	• *Solve a variety of problems based on the multiplication principle of counting. (Gr. 4, new)* • *Convert between expressions for positive rational numbers, including fractions, decimals, and percents. (Gr. 6, new)* • *Study the following topics related to vertex-edge graphs: Euler circuits, Hamilton circuits, the Traveling Salesperson Problem (TSP), minimum weight spanning trees, shortest paths, vertex coloring, and adjacency matrices. (Gr. 9-12 ,new)*
CA	64 changes (Gr. K-12, including new standards, additional words added to CCSS-M standards and movement of standards from one grade to another)	• *Identify the time (to the nearest hour) of everyday events (e.g., lunch time is 12 o'clock, bedtime is 8 o'clock at night). (Gr. K, new)* • *Tell and write time from analog and digital clocks to the nearest five minutes, using a.m.* ***and p.m. Know relationships of time (e.g., minutes in an hour, days in a month, weeks in a year).*** *(Gr. 2, phrase in bold added)* • *Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle. (Gr. 7 in CCSS-M; Gr. 6 and 7 in CA version of CCSS-M)*
CO	Many word changes, addition of personal financial literacy standards (Gr. K-12)	• *Identify two fractions as equivalent (equal) if they are the same size, or the same point on a number line. (Gr. 3, "Understand" changed to "Identify")* • *Know there is a Define the complex number i such that $i^2 = -1$, and show that every complex number has the form a + bi where a and b are real numbers. (Gr. 9-12, changed "Know there" is a to "Define the")*
IA	Thirteen standards added (two standards added at Gr. 2, eleven added at Gr. 9-12)	• *Use interviews, surveys, and observations to collect data that answer questions about students' interests and/or their environment. (Gr. 2)* • *Understand, analyze, apply, and evaluate some common voting and analysis methods in addition to majority and plurality, such as runoff, approval, the so-called instant-runoff voting (IRV) method, the Borda method and the Condorcet method.(Gr. 9-12)*
KS	Encourage additional emphasis on Probability/Statistics & Algebraic Patterning	
MA	*25 standards added (two at Gr. 1, two at Gr. 2, one at Gr. 4, one at Gr. 5, five at Gr. 6, two at Gr. 7, twelve at Gr. 9-12)*	• *By the end of Grade 2, know from memory related subtraction facts of sums of two one-digit numbers. (Gr. 2)* • *Solve problems that relate the mass of an object to its volume. (Gr. 6)* • *Use equations and graphs of conic sections to model real-world problems. (Gr. 9-12)*
NY	2 standards added (one at Gr. K and one at Gr. 1)	• *Develop understanding of ordinal numbers (first through tenth) to describe the relative position and magnitude of whole numbers (K)* • *Recognize and identify coins, their names, and their value. (Gr. 1)*

Table 3. States participating in PARCC and SBAC

PARCC		SBAC	
Alabama	Massachusetts (G)	Alabama	Nevada (G)
Arizona (G)	Mississippi (G)	California (G)	New Hampshire (G)
Arkansas (G)	New Jersey (G)	Connecticut (G)	North Carolina(G)
Colorado (G)	New Mexico (G)	Delaware (G)	North Dakota
District of Columbia (G)	New York (G)	Hawaii (G)	Oregon (G)
Florida (G)	North Dakota	Idaho (G)	Pennsylvania
Georgia (G)	Ohio (G)	Iowa (G)	South Carolina
Illinois (G)	Oklahoma (G)	Kansas (G)	South Dakota (G)
Indiana (G)	Pennsylvania	Maine (G)	Vermont (G)
Kentucky	Rhode Island (G)	Michigan (G)	Washington (G)
Louisiana (G)	Tennessee (G)	Missouri (G)	West Virginia (G)
Maryland (G)		Montana (G)	Wisconsin (G)
			Wyoming

"G" indicates role as governing partner.

> and [Virginia Department of Education] are ensuring that expectations for teaching and learning in Virginia schools are comparable to, or in some instances exceed, those of the voluntary national standards. (paragraph 1)

Likewise, in the 2011 draft revision of the Texas Essential Knowledge and Skills (TEKS), writers drew heavily from CCSS-M, in some cases using identical language. (See *The Commissioner's Draft of the Texas Mathematics Standards,* http://www.tea.state.tx.us/index2.aspx?id=2147499971).

In summary, while most states adopted CCSS-M without modification, a few states have chosen to augment or include clarifying examples or annotations. The extent of augmentation ranges from adding one or two standards at a particular grade level (e.g., Iowa) to movement of standards across grade levels and changes in wording (e.g., California). On the other hand, most states adopted CCSS-M as published, thereby adhering to the goal of "common" standards across states. However, in many states additional documents or materials were developed to support teachers as they transition to CCSS-M.

Collaborating on Assessments Aligned with the CCSS-M

Since the adoption of CCSS-M, states have joined and contributed to one or both of two state-led assessment consortia funded by the U.S. Department of Education—PARCC and SBAC. (See Table 3.) States contribute as a "governing" partner to a single consortium, or as a "participating" partner, where they monitor the work of both consortia but delay a decision regarding use of a particular consortia assessment. Both consortia are committed to developing technology-based adaptive mathematics assessments for students in grades 3-8 and high school. These assessments will report students' progress toward and attainment of the knowledge and skills required for college and career readiness as defined by CCSS-M.

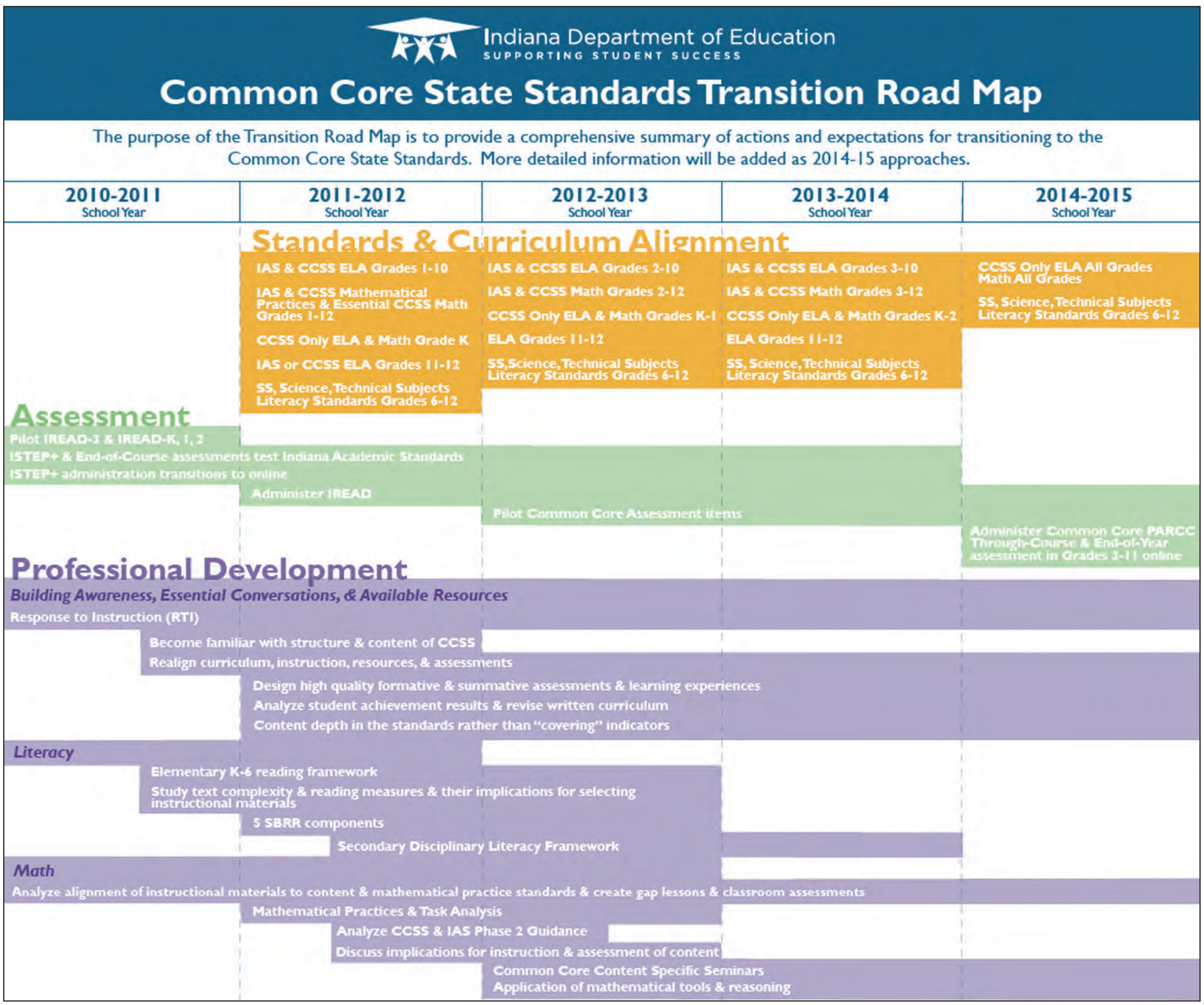
Information about the nature and extent of involvement in the assessment consortia is not readily available on many state departments of education websites. However, consortia websites indicate state level involvement in various consortia committees or work groups (e.g., PARCC Committees such as K-12 Leadership Team, Higher Education Leadership Team, and Technical Advisory Committee and SBAC Work Groups such as Accessibility and Accommodation, Item Development, and Test Administration). Perhaps the most notable contribution of states to the assessment consortia is the development of consortia assessment frameworks that are guiding the construction of assessments. These frameworks are available for public review (see http://www.parcconline.org/parcc-model-content-frameworks and http://www.smarterbalanced.org/smarter-balanced-assessments/) and will be used by providers who respond to a call to create elements of the consortia assessments via competitive bids.

The common, CCSS-M-aligned assessments are expected to be ready for full implementation in 2014-15. In the meantime, states are utilizing either their existing state assessment system or a modified version of their state assessment system that represents some attention to CCSS-M. In either case, many state departments of education have developed implementation timelines, presenting plans and deadlines for transitioning from the current state standards and assessments to CCSS-M. Based on a review of the state timelines for implementation of CCSS-M, we summarize here some features of the transition plans.

Transitioning to CCSS-M

Most states began their transition to CCSS-M by developing a "crosswalk document" that compared the current state standards to CCSS-M. The document provides a means for teachers to understand changes in student learning expectations and, thus, in instructional emphasis. In addition, some states developed "bridging documents" including timelines for transitioning from current standards to CCSS-M as well as recommendations for graduated implementation of CCSS-M (e.g., partial implementation of CCSS-M in some grades or moving some standards from one grade to another in preparation for the full transition). The transition timelines include specification of when teachers are expected to use CCSS-M, rather than current state standards, in determining the focus of their instruction. In some cases, the bridging plan also includes staged plans for professional development of teachers, and identification of the year in which state assessments will align with CCSS-M (e.g., Indiana's initial timeline is shown in Figure 1).

FIGURE 1: INITIAL *Indiana timeline for CCSS-M implementation*



Source: Retrieved November 14, 2011, from http://www.doe.in.gov/sites/default/files/curriculum/transition-road-map-implementing-common-core-state-standards1_0.pdf

The timelines for how and when states are transitioning to CCSS-M vary considerably. For example, Florida's timeline (see http://www.fldoe.org/arra/pdf/CCSSRolloutTimeline.pdf) includes a phased-in implementation of CCSS-M as follows:

- Gr. K in 2011-12;
- Gr. K-1 in 2012-13;
- Gr. K-2 in 2013-14;
- Full K-12 implementation in 2014-15.

In contrast, in Kentucky, "Teachers will begin to provide instruction related to the standards in the fall of 2011. Students will be assessed on the Common Core Standards beginning in the spring of 2012" (Kentucky Board of Education, Press Release, Feb. 10, 2010).

In addition to Kentucky, a few states began implementation of CCSS-M during 2011-12. For example, Arizona and Florida implemented CCSS-M in grade K; Arkansas, Nevada, New Hampshire, New Jersey, and Oregon in grades K-2; Mississippi in grades K-8; and Utah in grades 6 and 9. These states will continue to transition to CCSS-M in other grades in subsequent years.

Some state departments of education initially encouraged teachers to focus on implementing the *Standards for Mathematical Practice of CCSS-M*. For example, as shown in Figure 1, Indiana teachers were directed to focus on the standards for mathematical practice in the first phase of implementation (2011-12) in addition to implementing the mathematical content standards in Kindergarten. In other states, decision-making regarding implementation of CCSS-M is focused at the school district level, rather than the state level. For example, state officials in Tennessee encourage districts to choose when they will implement CCSS-M within the period 2011-14. In some cases, implementation of CCSS-M is dictated by state legislation or policy. For example, California will suspend the normal state-facilitated curriculum review cycle, delaying the development of a curriculum framework until July 2015.

As early as 2010-11, some states had already initiated professional development related to CCSS-M. For example, Kansas sponsored a series of one-week regional academies during the summer of 2011 focused on assisting teachers and administrators with preparation for the transition to CCSS-M. Other states (e.g., Louisiana) offered webinars for teachers and administrators.

A few state departments of education (e.g., Missouri, South Dakota, Utah) are sponsoring or partnering with others on the development of curriculum materials aligned to CCSS-M. In some cases, this work is intended to provide support for teachers until new CCSS-M-aligned textbooks are available and can be reviewed and purchased. In other cases, it responds to the need for particular kinds of materials. For example, the Utah Department of Education is partnering with *The Mathematics Vision Project* (http://www.mathematicsvisionproject.org/) on the development of high school materials aligned with the Common Core State Standards as organized within the *Integrated Mathematics 1* pathway (Appendix to CCSS-M).

Summary

The release of CCSS-M and its subsequent adoption has set in motion a massive effort across the nation to understand the new standards, assimilate CCSS-M into existing state structures, plan for implementation and, in some cases, begin implementation in classrooms. As summarized in this article, states are institutionalizing CCSS-M in various ways and are approaching implementation through state-led or localized district-led activities.

Based on a review of state department of education websites and communication with state department staff, we have summarized here activity at the state level in response to CCSS-M. However, it is not clear which components or how much of this effort and activity is penetrating to the district and school or teacher level. Additional research (e.g., district case studies) is needed in order to understand how this major policy initiative is playing out at all levels of the educational system.

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Standard Algorithms in the Common Core State Standards

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In a recent issue of the *Journal for Mathematics Education Leadership* comparing many state standards to the Common Core State Standards for Mathematics (CCSS-M), Reys and Thomas (2011) noted the following

> "[The] specific statement of the culminating standard for each operation in CCSS-M includes the expectation of use of 'the standard algorithm.' … However, a definition for 'the standard algorithm' is not offered. If the authors of CCSS-M had a particular standard algorithm in mind, it was not made explicit nor is an argument offered for why a particular (standard) algorithm is expected." (p.26)

The issue of standard algorithms was addressed in the Number and Operations in Base Ten (NBT) Progression document written by individuals involved in the creation of the CCSS-M as a narrative discussion of the learning progression of standards within a particular domain across grade levels. This progression document can be found at http://commoncoretools.me/category/progressions/. Questions about the standards and the learning progressions associated with them have also been discussed by authors of the CCSS-M at the http://commoncoretools.me web site. This article draws on the CCSS-M, the NBT Progression document, and the webpage dialogue with the authors of the CCSS to explore the question of what is meant by a standard algorithm.

For multidigit computation, the CCSS-M specifies a learning progression in which students develop, discuss, and use efficient, accurate, and generalizable methods based on place value and properties of operations. Students explain the reasoning used in a written method with visual models. Then, in a later grade, students move to using the standard algorithm fluently with no visual models. While the CCSS-M includes specific standards addressing the understanding of place value, in this article we focus only on the standards addressing multidigit computation, though it is vital to understand that an important goal of these computation standards is to deepen and extend place value concepts and skills.

The National Research Council report *Adding It Up* (Kilpatrick, et al., 2001) described many variations of algorithms that are used in the United States, and there have been analyses and discussions about which variations of these algorithms might be best used for at least the last century. Variations of algorithms also exist in other countries. For instance, Fuson and Li (2009) identified a number of variations of algorithms for multidigit addition and subtraction found in textbooks in China, Japan, and Korea. It is important to ask what is intended by the term *the x* of the CCSS-M chose to use the term *the standard algorithm* rather than *a standard algorithm*.

In a dialogue posted on April 25, 2011, CCSS-M lead author Bill McCallum suggested that the following explanation from the NBT Progression document would help address the question of what is meant by *the standard algorithm* or *a standard algorithm:*

> "In mathematics, an algorithm is defined by its steps and not by the way those steps are recorded in writing. With this in mind, minor variations in methods of recording standard algorithms are acceptable." (NBT, p13)

The NBT Progression document also defines a standard algorithm as follows:

> "Standard algorithms for base-ten computations with the four operations rely on decomposing numbers written in base-ten notation into base-ten units. The properties of operations then allow any multi-digit computation to be reduced to a collection of single-digit computations. These single-digit computations sometimes require the composition or decomposition of a base-ten unit." (p13)

Taken together, the NBT Progression document summarizes that *the standard algorithm* for an operation implements the following mathematical approach with minor variations in how the algorithm is written:

- Decomposing numbers into base-ten units and then carrying out single-digit computations with those units using the place values to direct the place value of the resulting number; and
- Using the one-to-ten uniformity of the base ten structure of the number system to generalize to large whole numbers and to decimals.

In the remaining portions of this article we identify variations in written methods for recording the standard algorithm for each operation, discuss what we believe could be considered minor variations in these algorithms, and suggest criteria for evaluating which variations might be used productively in classrooms. We also clarify the terms strategy, written method, and standard algorithm—all of which are used in the CCSS-M and the NBT Progression document —and suggest ways in which leaders in mathematics education can use the information in this article to help teachers and students understand and become fluent in base-ten computation.

Strategy, Standard Algorithm, and Written Method

The key NBT standards and related excerpts from the grade level introductions to the critical areas from the CCSS-M are given for multidigit addition and subtraction in Table 1 and for multidigit multiplication and division and all operations on decimals in Table 2. In all grades including Grade 1 students are to develop, discuss, and use efficient, accurate, and generalizable methods. The initial methods use strategies based on place value and properties of operations; these are related to written methods and the reasoning is explained using visual models (concrete models or drawings in Grades 1 and 2 and drawings/diagrams in Grades 4 and 5).

The word "strategy" emphasizes that computation is being approached thoughtfully with an emphasis on student sense-making. *Computation strategy* as defined in the Glossary for the CCSS-M includes special strategies chosen for specific problems, so a strategy does not have to generalize. But the emphasis at every grade level within all of the computation standards is on efficient and generalizable methods.

For each operation, as discussed above, there is a particular mathematical approach that is based on place value and properties of operations; an implementation of the particular mathematical approach is called the *standard algorithm* for that operation. To implement a standard algorithm one uses a systematic *written method* for recording the steps of the algorithm. There are variations in these written methods. Some of these variations are a little longer because they include steps or math drawings that help students make sense of and keep track of the underlying reasoning. Over time, these longer written methods can be abbreviated into shorter written methods that allow students to achieve fluency with the standard algorithm while still being able to understand and explain the method.

We have discussed above that standard algorithms rely on the particular mathematical approach of decomposing numbers into base-ten units and then carrying out single-digit computations with those units. They are efficient because they use place-value knowledge and single-digit computations that have already been developed. Because of the consistent one-for-ten structure across all whole number and decimal places, these algorithms thus generalize to large whole numbers and to decimals. As Bill McCallum says (April 29, 2011) about Grade 2: "Using three digits rather than two allows one to illustrate the iterative nature of the algorithms, and emphasize the fact that the base ten system uses the same factor, 10, for each rebundling of units into higher units." The standard algorithms are especially powerful because they make essential use of the uniformity of the base-ten structure. This results in a set of iterative steps that allow the algorithm to be used for larger numbers. For addition and subtraction, this is first visible for totals larger than 100.

Table 1. NBT Standards that Focus on Multidigit Addition and Subtraction and Related Grade-Level Critical Areas

GRADE 1 CRITICAL AREA (2) Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones.

Grade 1: Use place value understanding and properties of operations to add and subtract.

4. Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

GRADE 2 CRITICAL AREA (2) Students use their understanding of addition to develop fluency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations.

Grade 2: Use place value understanding and properties of operations to add and subtract.

5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

9. Explain why addition and subtraction strategies work, using place value and the properties of operations. [Explanations may be supported by drawings or objects.]

GRADE 3 CRITICAL AREA There is no critical area for multidigit computation.

Grade 3: Use place value understanding and properties of operations to perform multi-digit arithmetic.

2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

GRADE 4 CRITICAL AREA (1) Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. (This continues in Table 2.)

Grade 4: Use place value understanding and properties of operations to perform multi-digit arithmetic.

4. Fluently add and subtract multi-digit whole numbers using the standard algorithm.
[Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.]

Criteria for Emphasized Written Methods

In the past, there has been an unfortunate dichotomy suggesting that *strategy* implies understanding and *algorithm* implies no visual models, no explaining, and no understanding. In the past, teaching *the standard algorithm* has too often meant teaching numerical steps rotely and having students memorize the steps rather than understand and explain them. The CCSS-M clearly do not mean for this to happen, and the NBT Progression document clarifies this by showing visual models and explanations for various written methods for standard algorithms for all operations. General methods that will generalize to and become standard algorithms can and should be developed, discussed, and explained initially using a visual model. Given this emphasis on meaning-making, variations in ways to record the standard algorithm that support and use place value correctly should be emphasized. Given the centrality of single-digit computations in algorithms, variations that make such single-digit computations easier should be emphasized. Different written methods for

Table 2. NBT Standards that Focus on Multidigit Multiplication and Division and Related Grade-Level Critical Areas and on All Operations with Decimals

GRADE 4 CRITICAL AREA (1) Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop fluency with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate and mentally calculate quotients, and interpret remainders based upon the context.

Grade 4: Use place value understanding and properties of operations to perform multi-digit arithmetic.

5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

6. Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

GRADE 5 CRITICAL AREA (2) Students develop understanding of why division procedures work based on the meaning of base-ten numerals and properties of operations. They finalize fluency with multi-digit [addition, subtraction] multiplication and division. They apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They [develop fluency in these computations, and] make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of decimals to hundredths efficiently and accurately.

Grade 5: Perform operations with multi-digit whole numbers and with decimals to hundredths.

5. Fluently multiply multi-digit whole numbers using the standard algorithm.

6. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction (and between multiplication and division); relate the strategy to a written method and explain the reasoning used.

GRADE 6 CRITICAL AREA There is no critical area for multidigit computation.

Grade 6: Compute fluently with multi-digit numbers and find common factors and multiples.

2. Fluently divide multi-digit numbers using the standard algorithm.

3. Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Note: There are two glitches in the Critical Area for Grade 5. The words in brackets are not consistent with the standards themselves, so should be omitted. In 5.NBT.7 the words “and between multiplication and division” should follow “between addition and subtraction” and so were inserted there in parentheses. Also, the word *procedures* is used beginning in Grade 4 rather than the word *methods* used in earlier grades. No change in meaning is intended (e.g., Grade 4 “procedures” are not more rote than are Grade 2 “methods”).

recording standard algorithms vary in three additional features. First, they always involve different kinds of steps, e.g., ungrouping (borrowing) to be able to subtract and the actual subtracting. These kinds of steps can alternate or can be completed all at once. Variations in which the kinds of steps alternate can introduce errors and be more difficult. Second, variations can keep the initial multidigit numbers unchanged, or single-digit numbers can be written so as to change (or seem to change) the original numbers. The former variations are conceptually clearer. Third, many students prefer to calculate from left to right, consistent with how they read numbers and words, so variations that can be undertaken left to right are helpful to many students.

Multidigit Addition

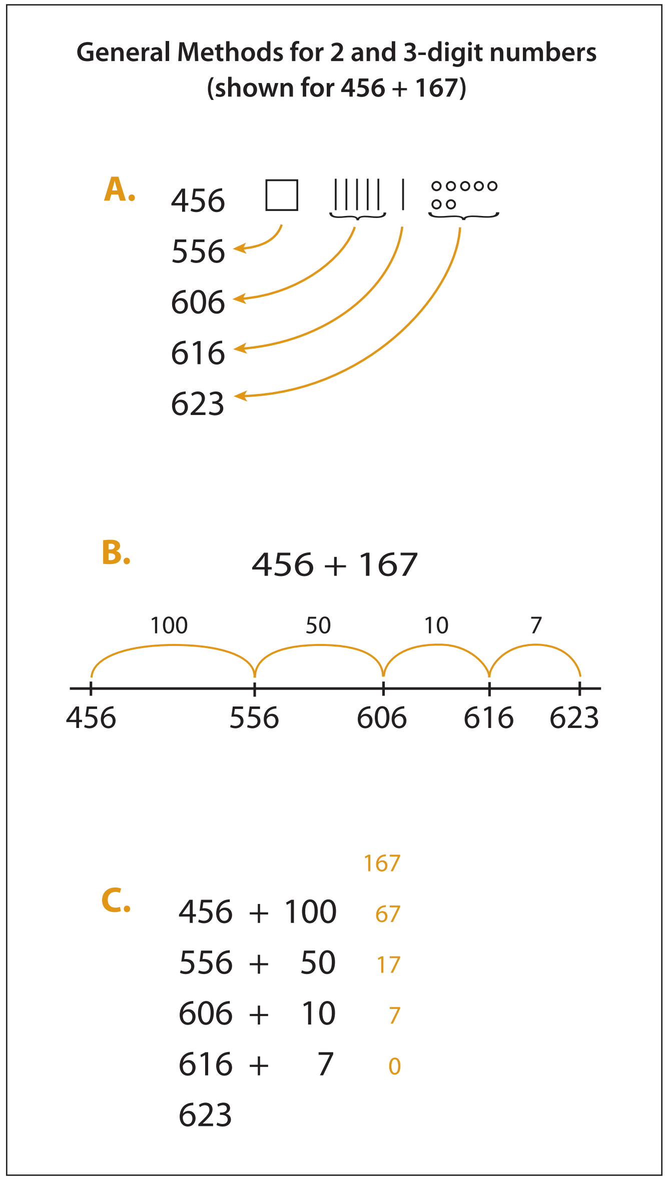
Now let us examine the examples of written methods for the standard algorithm given in the NBT Progression document and some other common strategies and consider the issues of how much variation in a written method is sensible and which variations might be emphasized. In the research literature, two approaches to multidigit addition and subtraction have been identified. (See Fuson, 1990, 1992; Fuson, et al., 1997; Verschaffel, et al., 2007):

a. decomposing into base-ten units (also called collection-based or split), which is the approach of the standard algorithm and

b. beginning with one undecomposed number (also called sequence or jump).

Figure 1 shows a count or add on approach that begins with one undecomposed number. With written methods for such an approach, one needs to keep track at each step of how much of the second addend one has already counted or added on. Two of the several variations of such keeping track methods using visual models are shown as Methods A and B. Variations of what is counted or added on first are possible (e.g., one might add on 4 to make 456 be 460), and the number of steps involved can vary. Method C keeps track numerically rather than with a drawing. Other variations are shown and discussed in Fuson, et al. (1997) and in Verschaffel, et al. (2007) and in NCTM (2010, 2011). These written methods are general methods for all 3-digit numbers but they are not practical for larger numbers. Even with these 3-digit numbers, one can see that it is a bit tricky to keep track of which places in the increasing total change at each step and perhaps even to notice explicitly that one is adding on like units. These methods are easier for 2-digit numbers and may arise

FIGURE 1. *Multidigit Addition Methods that Begin with One Undecomposed Number (Count or Add On)*



when students extend counting on methods with single-digit numbers.

Written methods for the standard algorithm that are generalizable to larger numbers and to decimals use single-digit computations of place-value units and are given in Figure 2. A drawing that could be used to direct or make sense of any of these written methods is given in the top row. These place-value drawings can help direct the steps in these methods by stimulating adding like units and composing as needed, as specified in the CCSS-M: *one adds hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose tens or hundreds.*

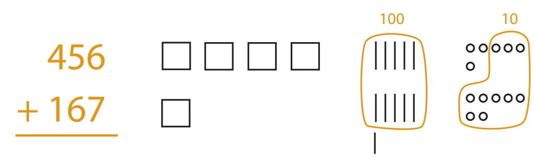
Writing the addends above each other helps students add like units for the methods that decompose base-ten units. Students can begin to use any of these methods without drawings whenever they no longer need these visual models, though they might make such drawings when explaining their method to classmates who might still need such a sense-making support. Dropping the drawings could begin to happen at Grade 1 or 2, but it definitely should be happening at Grade 3 so that students can focus on extending their method to larger numbers in Grade 4. Drawings could be used initially in Grade 4 especially for thousands, but they do not have to be used.

The expanded notation Method D that showed the sum of each place value can be extended to 1,000,000 but it could be difficult to keep all of the place value columns straight. This is an example of a written method that shows extra steps (helping steps) that can be very useful when students are developing understanding. But its steps can be collapsed into one of the other methods E, F, G as students move to larger numbers.

Which written methods meet more criteria of methods that should be emphasized? Method D is the only method that can be undertaken from the left (as well as from the

FIGURE 2: *Multidigit Addition Methods that Decompose into Base Ten Units*

Place value drawing for all methods



General methods for 2- and 3-digit numbers

```
D.   456      E.   456      F.  5 6       G.  1 1
   + 167         + 167          456           456
   -----         --1-1-       + 167         + 167
     500           623        -----         -----
     110                        623           623
      13
   -----
     623
```

Methods E, F, and G generalized to 6-digit numbers

```
E.    456,789     F.  5 6  8 9        G.  1 1  1 1
    + 167,189         456,789             456,789
    --1-1--1-1-     + 167,189           + 167,189
      623,978       ---------           ---------
                      623,978             623,978
```

Note: Methods E, F and G are all variations in the standard algorithm, but Method E is conceptually clearer and easier.

right) so this is an advantage initially when building understanding. Method D also shows the place values explicitly, so this also makes it clearer that one is adding like multiunits. Method E has several advantages, especially compared to Method G, and supports place value understanding and use by:

- making it easier to see the teen sums for the ones (13 ones) and for the tens (12 tens), rather than separating these teen sums in space as in Method G so that it is difficult to see the 13 or the 12;
- allowing students to write the teen numbers in the usual order as 1 then 3 (or 1 then 2) instead of, as in Method G, writing the 3 and then "carrying" or grouping the 1 above;
- making it easier to see where to write the new 1 ten or 1 hundred in the next left place instead of above the left-most place (a well-documented error that arises more with problems of 3 or more digits and is easier to make when one is separating the teen number as in Method G); and
- making it easier to write the new 1 on the line above exactly the correct (next left) column; when one writes the 1 above the addends in Method G the 1 is spatially separated farther.

It is easier to carry out the single-digit additions with Method E because you just add the two larger numbers you see and then increase that total by 1, which is waiting below. In Method G, students who add the two numbers in the original problem often forget to add the 1 on the top. Many teachers emphasize that they should add the 1 to the top number, remember that number and ignore the number they just used, and add the mental number to the other number they see. This is more difficult than adding the two numbers you see and then adding 1. Method F adds the 1 into the top number instead of writing it above. This makes it easy to add the two numbers that are there, but some students get this method confused with subtraction because you are crossing out a number in the top. In Methods F and G, you change the problem by modifying the top number. With Method E, the two multidigit addends and the sum are all in their own spaces, which is conceptually clearer.

Method E meets more criteria as an emphasized method, so it can be introduced in Grade 1 (if no students develop it) along with Method D, which can move from the left and helps students see the values of the places. Even though Method G has many disadvantages, many parents are familiar with it, so it is useful to discuss and explain it in the classroom and relate it to other methods. Method F makes the addition easy to carry out, but it does change the problem and is easily confused with subtraction, so it might be better to avoid it unless students develop it themselves.

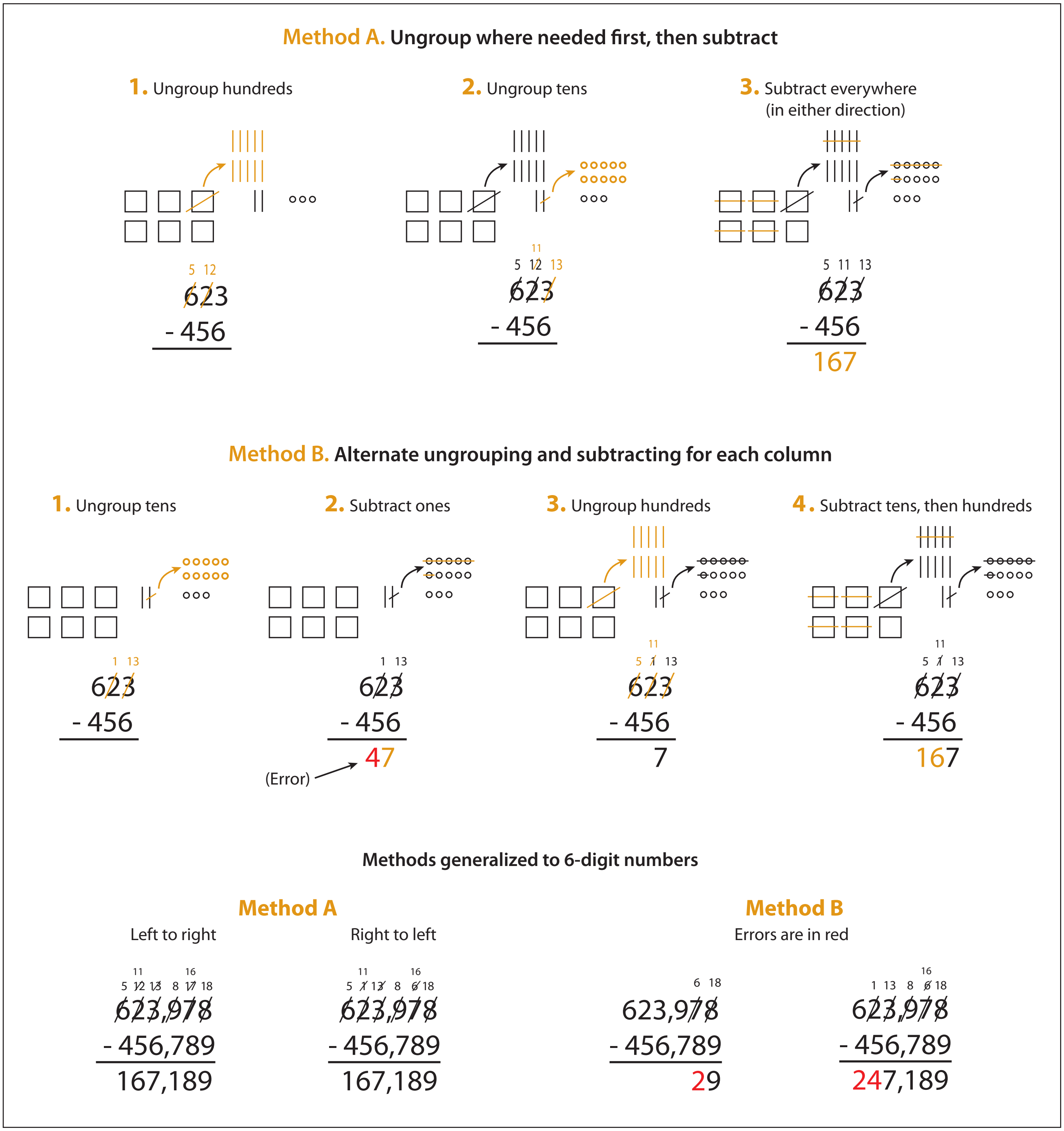
Multidigit Subtraction

Multidigit subtraction approaches that begin with one undecomposed number can involve counting or adding on up to the known total or can involve counting or subtracting down. Written methods for the former approach would look like the methods in Figure 1, but the sum is known and the unknown addend is being found. When finding the unknown addend, one monitors when one has reached the known sum and then finds how many were added/counted on. More variations of written methods that begin with one undecomposed number are shown in Fuson, et al. (1997), in Verschaffel et al. (2007), and in NCTM (2010, 2011). But as with addition, these methods are not practical for larger numbers, and some students find them difficult even for 3-digit numbers.

Written methods for the standard subtraction algorithm use the approach of decomposing into base-ten units and are shown in Figure 3. For subtraction, you need to check the number of units in the top number for a given column to see if there are enough of those units to subtract from (e.g., *Is the top number greater than or equal to the bottom number?*). If not, you need to get more of those units by ungrouping one unit from the left to make ten more of the units in the target column. All of these "checking and ungrouping if needed" steps can be done first, either from the left or from the right. Then all of the subtracting can be completed either from the left or from the right. (These subtractions can actually be completed in any order, but going in one direction systematically creates fewer errors). This taking care of all needed ungrouping first is shown as Method A with math drawings for a 3-digit example and then without drawings for a 6-digit number at the bottom to show how it generalizes. Students can stop making drawings as soon as they understand and can explain the steps.

Ungrouping from the left and from the right are shown for the 6-digit example. The only difference between these is in columns with two new units shown for a column.

FIGURE 3: *Multidigit Subtraction Methods that Decompose into Base Ten Units*



Separating the two major kinds of steps involved in multi-digit subtracting is conceptually clear and makes it easier to understand that you are not changing the total value of the top number when you ungroup. You are just moving units around to different columns. Many students prefer to move from left to right, as they do in reading, and productive mathematical discussions can take place as students explain why they can go in either direction and still get the same answer.

The Method B variation involves following the same steps but alternating between ungrouping and subtracting.

Alternating steps is usually more difficult, and this method sets up the common subtraction error of subtracting the top from bottom number when it is smaller (e.g., for 94 – 36, get 62). Even when you know you should check and ungroup if needed, alternating steps prompts errors. For example, in the 3-digit number in Step 2 you have just subtracted 6 ones from 13 ones to get 7 ones. You look at the next column and see 1 and 5, and 4 pops into your head (if you are only in second grade). You write 4 and move left. In the 6-digit problem, the three errors that can be created by alternating ungrouping and subtracting in Method B are in red. Although this alternating method can be used for numbers of any size, it is not as easy or conceptually clear as Method A. For 2-digit numbers, the alternating Method B and non-alternating Method A are the same because there is no iteration of the steps.

This top-from-bottom subtraction error noted above has been very frequent in the past, partly because of the usual practice of introducing problems with no ungrouping (e.g., 78 – 43) in Grade 1 and only moving to ungrouping problems a year later, in Grade 2, after students had already solidified a subtraction method that seemed to work well. That subtraction method involved looking at a column and subtracting the two numbers you saw there regardless of their relative size. It is our hope that the CCSS-M will result in the elimination of this common textbook practice because no general 2-digit subtraction methods (with or without ungrouping) are included in the Grade 1 standards. Therefore, in grade 2, subtraction with ungrouping can be addressed first so students learn to check from the beginning to see if they need to ungroup each column. This initial understanding of subtracting as possibly needing ungrouping, combined with reduced use of the alternating method, may greatly increase understanding of subtraction methods and contribute to a greater number of correct answers.

There are also strategies that only apply easily to some numbers. For example, for 98 + 47, a student may think, "I give 2 from the 47 to the 98 to make it 100, and then the 47 is 45, so I add 45 to 100 which is 145." This strategy recomposes both numbers to make a particularly easy addition. The subtraction counterpart to this strategy is more difficult for students (and some teachers) because one must know and remember to keep the difference for the original and the new problem the same: 145 – 98 has the same difference as 147 – 100, which is 47. Exploration of such limited strategies can support understanding of relationships between addends and sums in addition and subtraction, but such work cannot replace the extensive time needed to develop understanding of written methods for the standard algorithm and moving them to fluency. Also, it is not the case that learning the standard algorithm prevents students from discussing good strategies for very special cases, such as 398 + 427. Standard algorithms and special strategies are mathematical tools that students can learn to apply strategically. Asking, *Can this method be used for all numbers of a given size or be extended to larger numbers?* is a good mathematical practice in the classroom.

Multidigit Multiplication

The key NBT standards for multidigit multiplication and division and all operations on decimals and the related excerpts from the grade-level introductions to the critical areas from the CCSS-M are in Table 2. Multidigit multiplication does not deal with as many places as multidigit addition and subtraction, where 6 places may be involved as opposed to 5-digit products, and the learning path is shorter. Multiplication moves from the first year (in Grade 4) where the approach of the standard algorithm is developed and explained using visual models (diagrams) to the second year (in Grade 5) where the approach of the standard algorithm continues to be deepened and then is used fluently.

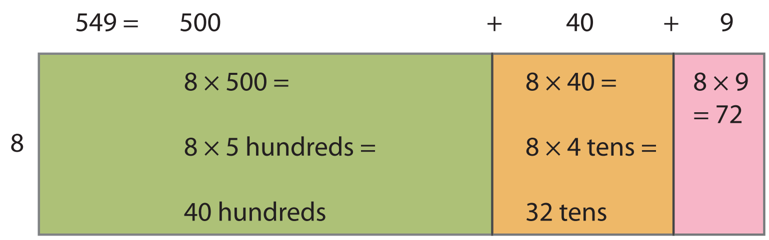
The major issue for multidigit multiplication is what to multiply by what and how the place values of the digits in the factors affect the place values of the partial products. An array or area model can help students understand these issues in terms of how the partial products are recorded. Figure 4 is modified slightly from the NBT Progression document and shows area models, distributive property equations addressing place value, and methods for recording the standard algorithm with 1-digit multipliers. These help students understand how each partial product comes from a multiplication of a kind of unit in one number times a kind of unit in the other number. The Methods A and B can be abbreviated to Method C where the partial products are written within the product space in one row rather than as separate rows that show the place values, but this method is more complex. Methods A and B are conceptually clearer.

Figure 5 shows how the place values in several methods for recording the standard algorithm for 2-digit by 2-digit multiplication relate to the place values in an area or array model and to each other. Methods D, F, and G write all four partial products, while Method E abbreviates the products of a given number into one row as did Method C. However, in Method E, partial products can be seen as diagonals written as for Method E in Figure 2 in addition (e.g., 24 ones is a small 2 in the tens place and a 4 in the ones place and likewise for the 54 tens, 12 tens, 27 hundreds). Writing these below allows students to see these products, and it puts all of the carries (regroupings) in the correct place. In another variation of this abbreviated method, shown in Figure 6 on the left as Method H, the 1 carried above the tens column is from 30 x 4 = 120, so it is actually 1 hundred and not 1 ten. It is confusing to have it in the tens column. Furthermore, having the carries above disconnects them from the rest of their product, so the steps and meanings of the digits can get confused. This method also alternates multiplying and adding, increasing its difficulty even further. This should not be an emphasized method but might be discussed if students bring it into the classroom. Method C also has a variation in which the carries are written above instead of below, which changes the problem, and makes it difficult to see the products because they are separated physically.

Multidigit multiplication can be a challenging visual-spatial task. Some students find it difficult to multiply without an area or array model. They prefer to make a quick area sketch, write the products inside, and then add up the

FIGURE 4: *Written Methods for the Standard Multiplication Algorithm, 1-digit × 3-digit*

Array/area drawing for 8 × 549



$$8 \times 549 = 8 \times (500 + 40 + 9)$$
$$= 8 \times 500 + 8 \times 40 + 8 \times 9$$

Method A:

Left to right showing the partial products

```
  549
×   8        thinking:
 4000        8 × 5 hundreds
  320        8 × 4 tens
   72        8 × 9
 4392
```

Method B:

Right to left showing the partial products

```
  549
×   8        thinking:
   72        8 × 9
  320        8 × 4 tens
 4000        8 × 5 hundreds
 4392
```

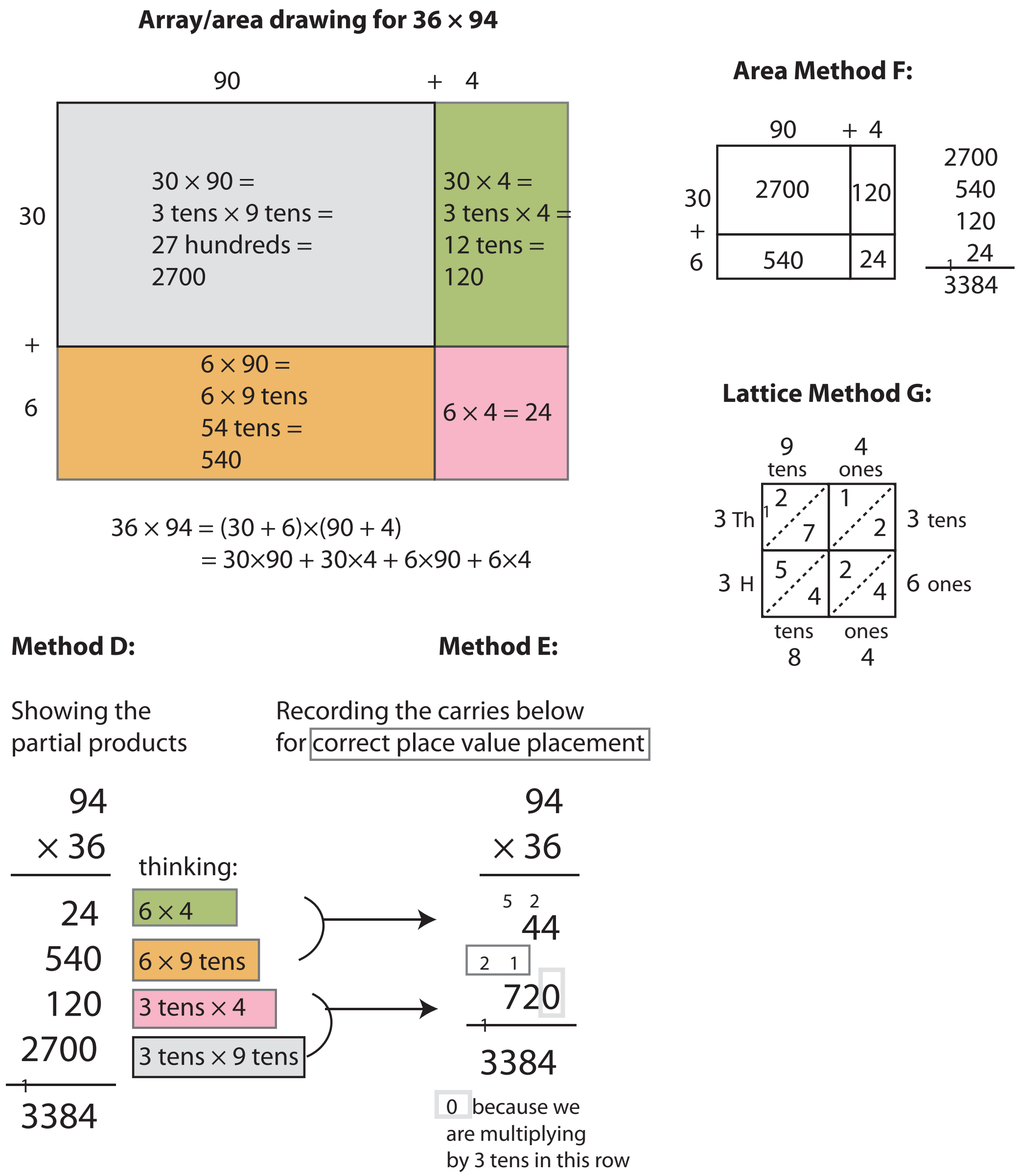
Method C:

Right to left recording the carries below

```
  549
×   8
 4 3 7
  022
 4392
```

Method A proceeds from left to right, and the others from right to left. In Method C, the digits representing new units are written below the line rather than above 549, thus keeping the digits of the products close to each other, e.g., the 7 from 8×9=72 is written diagonally to the left of the 2 rather than above the 4 in 549.

FIGURE 5: *Written Methods for the Standard Multiplication Algorithm, 2-digit × 2-digit*



Written Methods D and E are shown from right to left, but could go from left to right. In Method E, digits that represent newly composed tens and hundreds in the partial products are written below the line intead of above 94. This way, the 1 from 30 × 4 = 120 is placed correctly in the hundreds place and the digit 2 from 30 × 90 = 2700 is placed correctly in the thousands place. If these digits had been placed above 94, they would be in incorrect places. Note that the 0 in the ones place of the second line of Method E is there because the whole line of digits is produced by multiplying by 30 (not 3).

products outside as in Area Method F in Figure 5. Such a written method might be a little too long for fluency with the standard algorithm because it involves a drawing, though it shows place values clearly, can generalize, and is efficient. But it also seems better even in Grade 5 to allow some students to use Method F with accuracy than to use the more abstract partial products Method D if students are likely to make errors with this more abstract recording.

FIGURE 6. *Further Written Methods for the Standard Multiplication Algorithm, 2-digit × 2-digit*

Method H:

A misleading abbreviated method

```
  1  <------  From 30 × 4 = 120.
  2           The 1 is 1 hundred,
  94          not 1 ten.
× 36
-----
  564
 ¹282
-----
 3384
```

Helping Steps Method I:

$$
\begin{array}{rcl}
94 & = & 90 + 4 \\
\times 36 & = & 30 + 6 \\
\hline
30 \times 90 & = & 2700 \\
30 \times 4 & = & 120 \\
6 \times 90 & = & 540 \\
6 \times 4 & = & 24 \\
\hline
 & & 3384
\end{array}
$$

The lattice method of multiplication is hundreds of years old and has become popular in some instructional programs. In Figure 5 we can see how this Lattice Method G is related to the area model. The products of each digit in each factor are written within the area model squares. Diagonals are drawn within each such square to locate the tens and ones of each partial product coming from the multiplication of two single digits. These diagonals inside the whole area square have place values that move from ones to thousands from the bottom right to the bottom left and then up to the top left. These place values are derived from the patterns for multiplying place values (e.g., tens times tens is hundreds). We label the place values of the factors and diagonals in the lattice multiplication so that we can see how the place values in each partial product relate and align. If this method is used in the classroom, it is important to emphasize these place values, so that students understand what they are doing, and are not rotely memorizing a procedure. That is not a CCSS-M approach.

Method I in Figure 6 is a "helping step" version of Method D developed by a class of Grade 4 students from a low SES school. These students recognized that many of them were making mistakes using Method D, and they developed this method to help eliminate these mistakes. By writing out the tens and the ones in each factor, they could see the number of zeros, and thus use the patterns involving tens and hundreds more easily. (The partial products Method D and the Area Method F also show the place values in the factors). They wrote the biggest product first so they could correctly align the other products under it, which also has the additional advantage of showing the approximate size of the full product rapidly. They wrote out the factors of each partial product because some students were not systematic in the order in which they multiplied, and by writing the factors for each partial product, they could check on whether all partial products were included. These steps also supported student efforts to explain each step in the method, initially relating it to an area or array model but eventually omitting this model. Recording all of these steps may be too extensive for fluency with the standard algorithm, but students did stop recording particular steps as they no longer needed their support, thus moving toward the greater fluency of Method D.

Method D can be undertaken from left to right, as can Method I, so Method I can collapse to the left to right version of Method D. Area Method F and Method D are the conceptually clearest methods, and Method D is fast enough for fluency.

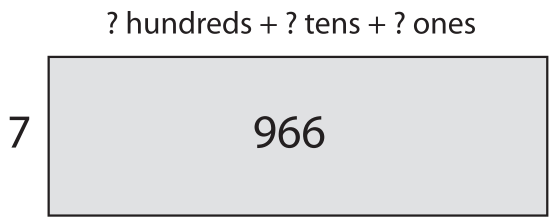
Multidigit Division

Like multiplication, multidigit division does not deal with as many places as does multidigit addition and subtraction, and again, the learning path is shorter. Developing fluency with multidigit division takes three years because students first develop and explain the approach of the standard algorithm with visual models for dividing by one-digit numbers in Grade 4 and then extend the approach in Grade 5 to dividing by 2-digit numbers where the difficulties of estimating complicate division. In Grade 6 the standard algorithm is used fluently for one- and two-digit divisors.

Array and area models can support understanding of strategies for division. Figure 7 shows Method A including both the recording of the numerical calculations and the area model that corresponds to those calculations as shown in the NBT Progression document. The full multiplier of the divisor at each step in Method A is written above the dividend so that students can see the place value and make a clear connection to the place values in the area

FIGURE 7: *Written methods for the standard division algorithm, 1-digit divisor*

Area/array drawing for 966 ÷ 7

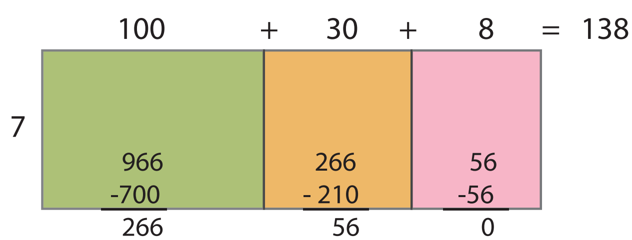



```
   ???
7 )966
```

Thinking: A rectangle has area 966 and one side of length 7. Find the unknown side length. Find hundreds first, then tens, then ones.

$$966 = 7\times100 + 7\times30 + 7\times8$$
$$= 7\times(100 + 30 + 8)$$
$$= 7\times138$$

Method A:




```
    8  ┐
   30  │ 138
  100 ─┘
7 )966
  - 700
    266
  - 210
     56
   - 56
      0
```

Method B:

```
   138
7 )966
  - 7
    26
  - 21
     56
   - 56
      0
```

Conceptual language for this method (all numbers below 966 are in square units):

Find the unknown length of the rectangle;
first find the hundreds, then the tens, then the ones.

The length gets 1 hundred (units); 2 hundreds (square units) remain.

2 hundreds + 6 tens = 26 tens (square units).

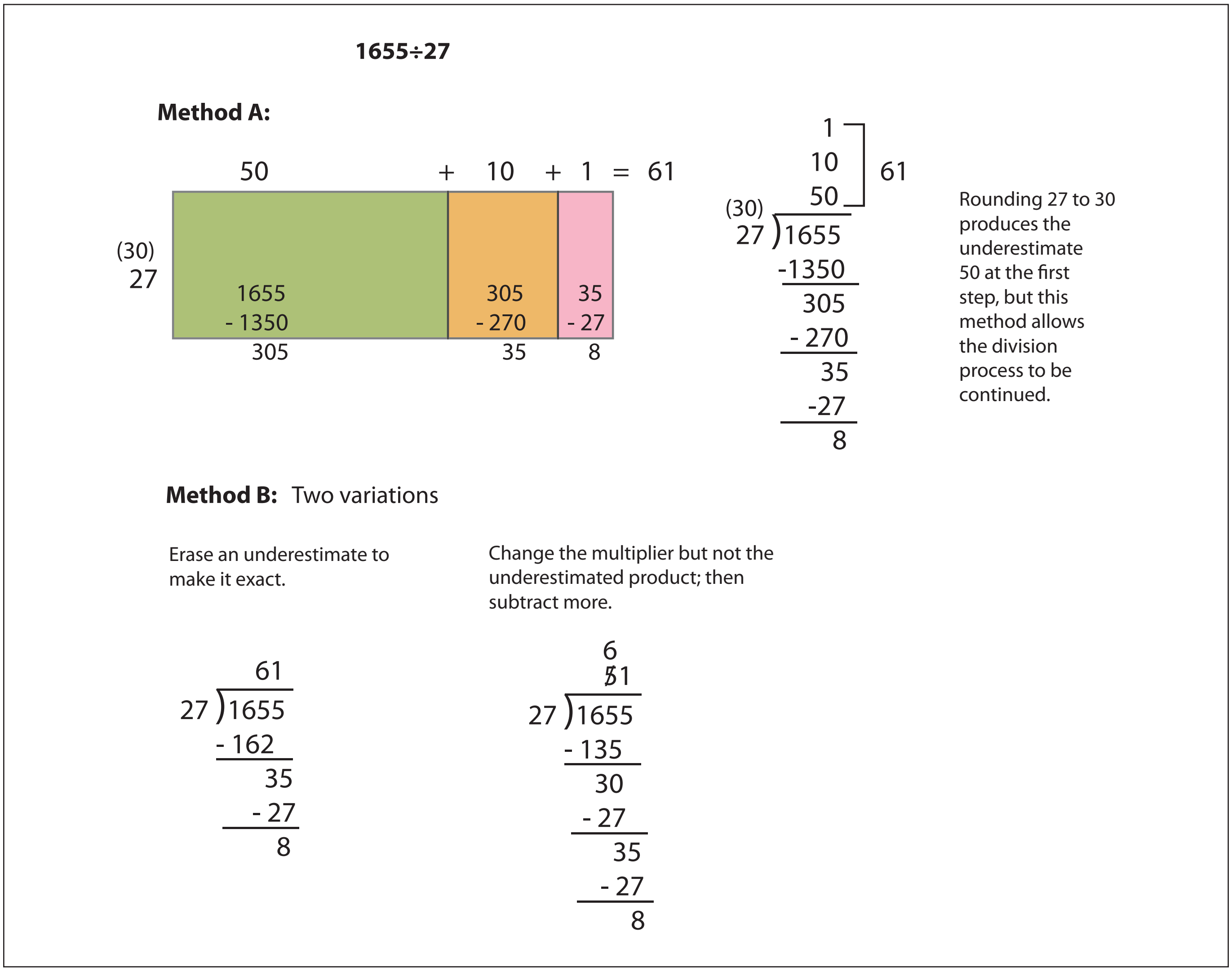
The length gets 3 tens (units); 5 tens (square units) remain.

5 tens + 6 ones = 56 ones (square units).

The length gets 8 ones; 0 remains.

The "bringing down" steps represent unbundling a remaining amount and combining it with the amount at the next lower place.

FIGURE 8: *Written methods for the standard division algorithm, 2-digit divisor*



model. The full product is written at each step, and the amount of the dividend not yet used is also written in full. After experience connecting it to a drawing, Method A can also be undertaken without a drawing as a standard algorithm when calculating quotients that are whole numbers or decimals.

Another written method for this standard algorithm is also shown as Method B in Figure 7. In this method, the zeros are not written in the multipliers or in the partial products within the problem. Digits are brought down within the problem one at a time. Method B makes it more difficult to make the connections to the meaning of the computation, and for that reason, we included conceptual language to communicate the underlying meaning of each step of the calculation. But this method does show clearly the single-digit calculations that are used, and these single-digit calculations are in their place-value locations, as indicated in Method A. Method B becomes important when a quotient has many places, for example, when explaining why decimal expansions of fractions eventually repeat (8.NS.1). Some students may be able to understand and explain Method B right away, and many students can move to it from strategies like Method A, because the move is small. There are also other written methods for the standard algorithm that are variations in between the two shown in Methods A and B. For instance, you could write the zeros on the top but not within the subtractions, or vice versa. Students might develop and use any of these variations.

Method A has a further advantage when dividing by 2-digit numbers because it can involve the use of reasonable estimates. The example shown in the NBT Progression document is depicted in Figure 8 and demonstrates how an estimate of the quotient can be used and then adjusted. Underestimates can be repaired in any place by subtracting another partial product for that place (and adding another rectangle to the area model when first using models). Such a simple repair is an acceptable written method for the standard algorithm. However, some introductions to division allow students to dramatically underestimate the quotient, and as a result, they then have many partial products (e.g., multipliers of 10 + 10 + 10 + 10 + 10) that are used to adjust the estimate. Many extra multipliers and partial products are not consistent with the fluency expectations of the standard algorithm, so students need to be encouraged to be brave and use a multiplier as close as possible to the largest multiplier, for the sake of efficiency. With Method B students cannot continue on from a low estimate. They need to be exact, which often means erasing their underestimate or overestimate. Of course, a student could leave the product and difference they already found, cross out their low multiplier, increase it by 1 or 2, and take away that partial product as another step as in Method A. (See the second variation in Figure 8). Note that overestimates are still best fixed by erasing and trying a lower partial quotient because repairs are difficult to carry out correctly.

Standard Algorithms for Operations on Decimals

The CCSS-M emphasize explaining operations for whole numbers using models that highlight the place value quantities and their roles in the operation. These understandings form the basis for operations with decimals in Grade 5 where students are also expected to explain these operations with models that highlight the place value quantities and their role in these operations. The mathematical approaches of the standard algorithms for whole number addition and subtraction that involve adding and subtracting like place value units, composing or decomposing where needed, also apply to the addition and subtraction of decimals. The lines of reasoning for whole number multiplication and division also extend to decimal multiplication and division. The extensions of these written methods for whole number computation to decimal computation are discussed in the NBT Progression document.

Roles of Leaders

Leaders in mathematics education have vital roles to play with respect to CCSS-M computation. They need to understand deeply and be able to explain the middle ground laid out in the CCSS-M so that they can lead teachers, parents, and administrators out of the sometimes deeply entrenched positions created during the “math war” years. This requires helping everyone understand that standard algorithms are to be understood and explained and related to visual models before there is any focus on fluency. The models help to build understanding for methods that highlight place value understandings and properties of operations. Full fluency is not achieved until subsequent years.

The real problem is with how the standard algorithms have been seen in the past—as fixed written methods learned rotely rather than as a mathematical approach based on a big idea that can be played out using various written methods. Furthermore, the recognition that easier or more meaningful methods can be chosen for emphasis in the classroom has often been lacking.

Leaders can help everyone understand that becoming fluent with the standard algorithm entails using a sensible written method that implements the mathematical approach of the standard algorithm. The mathematical approach of the standard algorithms is distinguished by a big mathematical idea—that multi-digit calculations can be reduced to single-digit calculations while at the same time attending to the placement of these digits by attending to their place values.

Leaders can highlight the power of this big mathematical idea and the importance of allowing written methods that empower students with sensemaking and a deeper understanding of the base-ten system and properties of operations as well as computational skill as they build fluency with standard algorithms. Although these written methods may look different from those that parents or teachers learned when they were in school, they nevertheless implement the deep mathematical ideas that are encapsulated in the standard algorithms.

Leaders need to be able to explain, for each operation, how visual models help explain important aspects of the place value and properties of operations in each of the written steps of the standard algorithm. They need to understand different variations in how to record these steps, as discussed in this paper and in the NBT Progression document. They need to be able to compare and discuss different variations, with attention to possible advantages and disadvantages of each variation at different points in a student’s learning. They need to be able to explain methods that may be brought into classrooms by students that may provide yet an additional variation that makes sense mathematically. They should also know ways to connect written methods to more advanced notation that uses place value and properties of operations, such as in this string of equations:

$$\begin{aligned} 8 \bullet 549 &= 8 \bullet (500 + 40 + 9) \\ &= (8 \bullet 500) + (8 \bullet 40) + (8 \bullet 9) \\ &= (8 \bullet 5) \bullet 100 + (8 \bullet 4) \bullet 10 + 8 \bullet 9 \end{aligned}$$

They should recognize that the expression on the last line encodes the approach of the standard multiplication algorithm, that it could be evaluated from left to right or right to left (after evaluating each term), and that such work is much like multiplication with polynomials, which students will learn in later grades. Such equations do not need to be used by students, but they can be helpful for teachers in seeing the big idea in action, and being able to relate it to mathematics that will come later for their students. Leaders need to use all of these understandings to support teachers, students, parents, and administrators in the development of their own understandings.

Above all, leaders need to help others see this CCSS-M conceptual approach to computation as deeply mathematical and as enabling students to make sense of and use the base ten system and properties of operations powerfully. How the regularity of the mathematical structure in the base ten system can be used for so many different kinds of calculation is an important feature of what we want students to appreciate in the elementary grades. The relationships across operations are also a critically important mathematical idea. The CCSS-M focus on understanding and explaining such calculations, with the support of visual models, enables students to see mathematical structure as accessible, important, interesting, and useful. This is the value of including the meaningful development of standard algorithms in the CCSS-M.

Conclusion

Various written methods that reflect the approach of the standard algorithm for each operation, as well as other general approaches, have been shown and discussed in this article. Some written methods are easier to understand or carry out and therefore should be introduced to students. Other written methods may be introduced by particular programs for various reasons. All variations are interrelated, and it is important for variations in written methods to be explored and discussed by students. Discussing and relating and explaining variations is also an important use of the Standards of Mathematical Practice found in the CCSS-M.

Our examples have attempted to carve out a middle ground for acceptable written methods that reflect the core mathematical approach of, and therefore can be considered to be, standard algorithms. We believe that methods with steps that show the crucial components of the standard algorithms (e.g., showing four partial products for a 2-digit x 2-digit multiplication) are acceptable versions of the standard algorithms but methods with many extra steps that make them less efficient are not acceptable versions.

We also believe that variations with many extra steps can be pedagogically useful as students progress toward the standard algorithms because they help students see and discuss the place value units and the properties of operations, with the extra "helping steps" being dropped by individual students as they progress toward fluency. However, because the CCSS-M emphasize understanding and explanation as the basis for moving to fluency, and some students may be moving more slowly along their learning trajectory than others, some students may continue to use visual models or a more extensive written method for the standard algorithm for some extra period of time.

Finally, we believe it is important for there be time and space in different instructional programs for the development of different written methods that support understanding and the development of fluency with the standard algorithms. These approaches and methods should be related to visual models that reflect the mathematical approach of the standard algorithm, and when written without visual models, can be extended to larger numbers and to decimals in later grades. These approaches and methods should also be able to be written systematically in ways that are clear and not misleading. We also believe that even with the richness of materials in some of our currently existing instructional materials, we may not yet know of some written methods that can also be useful to explore as we address the expectations of the CCSS-M. For this reason, we will have ongoing discussion of advantages and disadvantages of various written methods at the Mathematics Teaching Community, https://mathematicsteachingcommunity.math.uga.edu/ under the standard-algorithms tag.

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Challenging Courses and Curricula: A Model for All Students

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Introduction

The Common Core State Standards (CCSS) have been adopted by 45 states and the District of Columbia as of May 2012. The Standards for Mathematical Content, which describe the content to be taught at each grade level, have received much attention by state boards of education, school districts, administrators and teachers. The Standards for Mathematical Practice describe "varieties of expertise that mathematics educators at all levels should seek to develop in their students," including sense-making, reasoning, perseverance, and communicating mathematical arguments, and while these standards are also vitally important, they have received less attention.

Teachers and administrators in the Greater Birmingham Mathematics Partnership (GBMP)[1] believe that the Standards for Mathematical Practice have received less attention because: (1) teachers and administrators do not understand what some of the mathematical practices are trying to describe; (2) many teachers were taught in traditional lecture style and have never experienced learning in an environment focused on developing the mathematical practices (Mayer, Cochran, Mullins, Dominick, Clark, & Fulmore, 2011); (3) teachers struggle to envision what classrooms would look like where students learn content through engaging in the Standards for Mathematical Practice; and (4) many administrators and teachers focus on the Standards for Mathematics Content as the way to raise test scores and see the Standards for Mathematical Practice as less essential.

Early in the GBMP project, partners collaborated to define "Challenging Courses and Curricula" and this definition has shaped professional development model that, for the past seven years, has promoted classroom instruction consistent with the Standards for Mathematical Practice across the K-12 grade levels and at the undergraduate and graduate levels as well.

When teachers and administrators refer to "challenging" mathematics courses, they are often referring to only the most advanced coursework available (such as a calculus course taken in high school) or to an accelerated track of courses (such as an algebra course taken in 7th grade). A different conception of challenging courses was developed by GBMP with the support from the National Science Foundation Math Science Partnership program and also appears in the literature (US Department of Education, 2008; US Department of Education, 1997). GBMP's definition for challenging mathematics courses asserts that all courses can and should be challenging for the students who take them and should result in students who develop expertise with the Standards for Mathematical Practice. In this article we define challenging courses and provide examples of classroom practice guided by this definition.

[1] Award #EHR-0632522

The GBMP project believes that challenging courses and curricula (1) help students deepen their knowledge of the big ideas in mathematics; (2) promote student inquiry and reflection; (3) support the development of productive disposition; and (4) foster articulate written and oral communication. We also recognize that aligned assessment practices positively impact these four overarching goals.

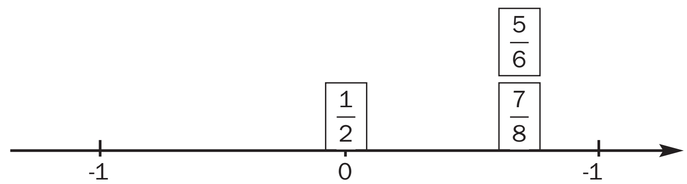
In our project, we are seeing classrooms where students are highly engaged in solving complex mathematics tasks, where students make sense of the mathematics they are doing, and where "talking mathematics" is the norm. All students are engaged but no student is held back from taking the mathematics as far as possible. In these classrooms, teachers think of mathematics as a sense-making discipline and help students make connections between and among seemingly unrelated mathematical ideas rather than viewing mathematics as sets of isolated skills and domains. What we see is consistent with our definition of challenging courses and curricula. We describe below the classroom environment and instructional practices found in these contexts.

Classroom Environment and Instructional Practices in Challenging Courses

1. BIG MATHEMATICAL IDEAS

In challenging courses, students investigate a coherent collection of problems organized around big mathematical ideas. Rather than focusing on isolated skills on an accelerated timeline, challenging courses focus on going deeply into the mathematical study of a few big ideas. In short, we fully appreciate the seemingly contradictory notion that by teaching fewer mathematics topics, but teaching them more thoroughly, learners will come to understand more mathematics and understand it as a fabric of connected and related ideas. This is consistent with the CCSS that emphasize learning critical content in depth.

In a challenging course, a whole class problem might be used to launch an investigation of some of the big mathematical ideas of fractions such as comparing and ordering, defining the whole, equivalence, and magnitude. These problems are often selected based on their potential to build understanding and reveal misconceptions. On one visit to a challenging classroom, we observed the teacher starting with a number line from -1 to +1 on the board. Students were asked to discuss with partners how they would order the following fractions: 1/2, 1/4, 7/8, 5/6, 2/4, 1/3, 1/5, and 3/5. After the discussion, each partner group placed one fraction on the number line. After all fractions were placed, the teacher asked students to discuss whether they agreed with the placement of the fractions and why. During the ensuing whole group discussion, the following big ideas and misconceptions emerged.



One student put 1/2 at 0 with the justification that 1/2 is halfway between -1 and +1. Another student said he thought 1/2 should be placed between 0 and 1 because 1 is the whole, and 1/2 is half of the whole, like half of a candy bar. In response to these ideas, and to focus students' attention on defining the whole, the teacher asked if 1/2 could be placed at both places.

Two partner groups argued that 7/8 and 5/6 were equivalent and should be at the same place on the number line because they were both one part away from the whole. Other students disagreed because 1/8 is smaller than 1/6 and so 7/8 is closer to one.

Throughout this lesson, students were developing Standards for Mathematical Practice (MP) including the following:

- Make sense of problems and persevere in solving them (MP 1);
- Reason abstractly and quantitatively (MP 2);
- Construct viable arguments and critique the reasoning of others (MP 3); and
- Attend to precision (MP 6).

2. INQUIRY AND REFLECTION

GBMP's conception of challenging courses is based on the belief that coming to know and understand important mathematical ideas takes time and that learning occurs through a process of inquiry and reflection. We view confusion—the cognitive dissonance that accompanies "not knowing"—as a natural and even desirable part of the process of constructing new knowledge. Challenging courses provide opportunities for students to struggle with problems, to find their own ways of solving them, and to recognize that there is usually not just one way to solve a problem. The dilemma for teachers is that they were often taught that a teacher's job is to teach how to best solve problems by giving clear explanations of each step to take in the solution. We have learned, however, that this natural inclination to want to put confusion to rest, and to "help" those who are struggling, is often counterproductive when it comes to developing mathematical understandings and productive dispositions.

We want to clarify our use of the word "confusion" and not leave the impression that we view all confusion as desirable. Some kinds of confusion need to be cleared up, especially when "social knowledge" is involved. For example, the use of a symbol may need to be explained or the language used in posing a problem may warrant clarification. But we have come to believe that teaching by telling rarely leads to deep mathematical understandings or productive mathematical dispositions. When students ask for help, teachers interact with them in ways that do not direct their thinking, listening to their thinking and asking probing questions in order to help students find their own ways through the problems.

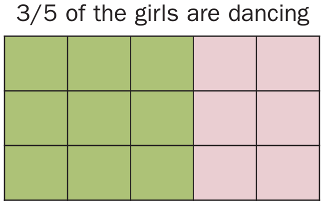
To illustrate, we describe an observation of students in a middle school classroom investigating the following Square Dance problem:

> For the first dance at the school square dance, 2/3 of the boys danced with 3/5 of the girls. What fraction of the students were dancing?

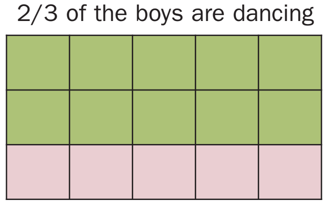
We recommend that you stop and think about this problem before reading on.

Students worked in small groups using color tiles to represent and make sense of the problem. Initially, one group thought they had a solution, but it involved finding a common denominator. They confronted the confusion that 9/15 + 10/15 = 19/15 is more than 100% of the students. Another group created the following diagram and said that 3/5 of the girls are dancing with 2/3 of the boys so 19/30 of the students are dancing.

3/5 of the girls are dancing

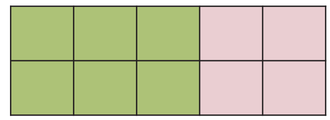


2/3 of the boys are dancing

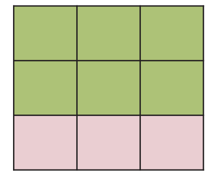


They confronted the confusion that one dancing boy did not have a partner. Eventually these groups wrestled their way out of their confusion and found a geometric solution that made sense to them. Using the diagram below, they argued that 3/5 of the girls were dancing with 2/3 of the boys, so 12/19 of all the students were dancing.

3/5 of the girls are dancing



2/3 of the boys are dancing



Another group attacked the problem algebraically and reasoned that $\frac{3}{5}I = \frac{2}{3}D$ where ***G*** is the number of girls and ***B*** is the number of boys. This group faced confusion about what to do next and made several unsuccessful attempts, eventually reasoning their way to the following solution that made sense to them. Since $\frac{3}{5}I = \frac{2}{3}D$, the number of boys is 9/10 times the number of girls, $D = \frac{9}{10}I$. Therefore, the fraction of students dancing is:

$$\frac{\frac{3}{5}I + \frac{2}{3}D}{I + D} = \frac{\frac{3}{5}I + \frac{3}{5}I}{I + \frac{9}{10}I} = \frac{\frac{6}{5}I}{\frac{19}{10}I} = \frac{12}{19}.$$

In solving this problem, students were modeling with mathematics (MP 4) in addition to addressing MP 1, 2, 3, and 6.

3. PRODUCTIVE DISPOSITION

Challenging courses are designed with the understanding that learning mathematics involves hard work. Even students who are confident in their mathematical content knowledge often encounter disequilibrium when they are asked to see problems in multiple ways or to solve a problem where the solution path is not immediately obvious to them. All students, no matter their level of competence or confidence, are engaged with mathematical problems that demand perseverance. Students learn what it means to struggle and to experience the satisfaction of finally solving a problem or understanding a mathematical idea. Students come to know that the degree of satisfaction or exhilaration they experience in solving a problem is often directly proportional to the amount of struggle and effort expended.

Challenging courses foster a productive and supportive learning community. Students come to care about each other's learning. They learn that in trying to understand the thinking of others they understand mathematics at a deeper level themselves. They learn how to ask for help by seeking guidance but not answers and they learn how to help other students without doing the mathematical thinking for them. Rather than rescuing students, teachers interact with students in ways that build more powerful mathematical understandings and dispositions that diminish the need for future rescue. Their goal is to help students become autonomous learners.

As an example, we visited a third grade classroom in which students were exploring whether halving and doubling was a strategy that would always work for multiplication. Students had noticed that to find the answer for a multiplication problem, you could halve one factor and double the other factor, and it would still give the same product (e.g., 5 × 18 = 10 × 9 = 90). One group of students discussed that this strategy was good for working with even numbers, but it wouldn't work with two odd numbers. Another student said that if the strategy was going to work, it would have to work in all cases, so let's see if it works with 7 × 7. The teacher heard this group discussion and knew that this would be a messy problem, but instead of stopping the students or suggesting an easier problem, she encouraged them to give it a try.

Alethia: 7 × 7 = 49; double 7 to get 14, and what's half of 7?

Mark: You can halve 6 to get 3 and half of 1 is ½, so half of 7 is 3½.

Shandra: So how do we multiply 3 ½ × 14?

Alethia: 3 × 14 = 42, and half of 14 is 7, and 42 + 7 = 49.

Students: It works! Let's see if we can do it again!

Undaunted, the students proceeded to investigate the problem by halving 3 ½ and doubling 14 (1¾ × 28). The students reasoned their way through this by computing 1 × 28 = 28; ½ × 28 = 14; ¼ × 28 = 7, and 28 + 14 +7 = 49, which led to cheers and applause at their own effort. The point of this example is not that this group of students figured out how to multiply a mixed number by a fraction (which is not a third grade standard), but that students were exploring properties of multiplication (which is a third grade standard) in an environment that encouraged them to ask their own questions and to persevere in finding the answers. The teacher also understood it was important to ask this group two questions: (1) Will this strategy always work? and (2) When would this be an efficient strategy?

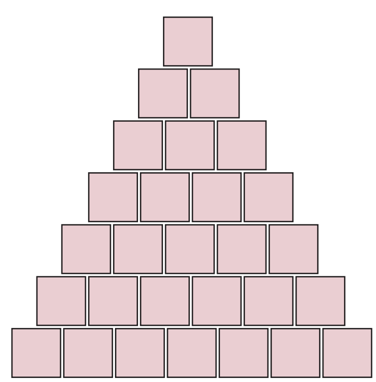
This vignette illustrates that the teacher valued investigation of mathematical ideas and believed students were capable of solving difficult mathematical problems. These 3rd graders believed that mathematics is supposed to make sense and they persisted in their sense-making process. They knew from experience that rich mathematical problems rarely have instant answers and so they were willing to persevere in reasoning through a challenging and unfamiliar problem. While this discussion provides opportunities to develop numerous Standards for Mathematical Practice, it particularly addresses perseverance in solving problems (MP 1) and looking for and making use of structure (MP 7).

4. COMMUNICATION

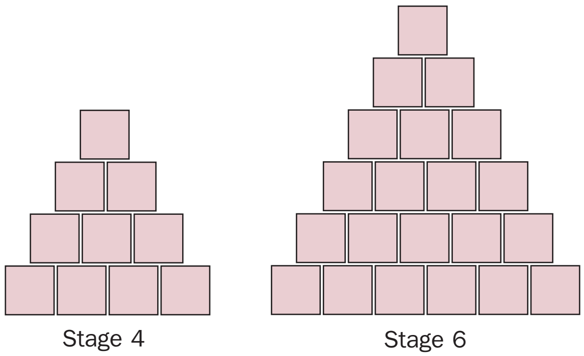
Talking and writing mathematics is the norm in challenging courses. Communication of mathematical thinking occurs in small groups as students work together to make sense of problems and during whole class processing of their thinking. An essential element of whole class processing is establishing a safe environment in which all students and mathematical ideas are treated with respect. During processing, students volunteer to share their diverse ways of seeing and solving problems. As different solutions and various representations (geometric, verbal, numerical, and algebraic) emerge, students deepen their understanding by making connections among various representations and solution paths. Whole class processing is done with an eye toward clarifying the mathematics involved and learning to consider, value, question, and build upon each others' mathematical ideas.

To illustrate, we describe an observation in an algebra class processing the following Building problem:

> A few stages of an increasing pattern are shown below. How many tiles would it take to build Stage 10? What about any stage? (Richardson, 1984).



Stage 7

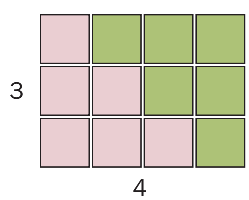


Again, you might want to stop and think about this problem before reading on.

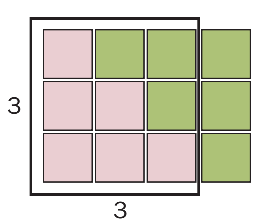
The teacher asked for volunteers and Patricia's hand went up.

Patricia: I was building stage 3, moving tiles around, and I realized I could "left justify" stage 3 to look like this (the diagram on the right below).

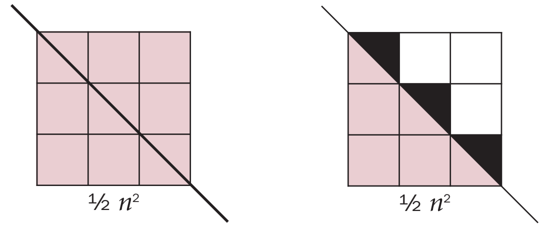
Then I put two copies of stage 3 together like this [see below]. Now it's easy to count that there are 3×4 tiles in all, but that's twice as many as I wanted, so there's really only (3×4)/2 tiles in stage three. For stage n there would be $[n\times(n+1)]/2$ tiles.



Xavier chimed in that he built the same arrangement of tiles as Patricia, but he saw a 3×3 square plus 3 more tiles. Then he also divided by 2. For stage n, his formula was $\frac{p^2 + p}{2}$.



Next JaMichal volunteered that he solved the problem by completing a square with color tiles, dividing the square in half, and adding back half of each tile on the diagonal for a result of $\frac{1}{2} n^2 + n/2$.



In addition to making sense of problems and communicating their ideas to others, students in this class exhibited MP 8 (Look for and express regularity in repeated reasoning). These students investigated the pattern for small values of n until they were able to determine a general formula. Mathematical Practice 5 (Use appropriate tools strategically) was also in evidence here. In this case, the tools in use were manipulatives, but in another problem the tool might be a protractor or a graphing calculator).

Conclusion

This article describes a broadly applicable vision for challenging mathematics courses. Whereas the common interpretation of "challenging" mathematics is relevant only for a small population of students enrolled in accelerated classes or enrichment programs, this definition applies to all mathematics courses and all students. The universality of the definition was one aim of the design—it is applicable not only to the K-12 classrooms described in our examples and to undergraduate and graduate courses and professional development institutes—but is also universal in another sense. Using this definition of challenging courses helps students develop mathematical practices that transcend any particular mathematics course. It builds their capacity to learn as much as it builds their knowledge of arithmetic, or geometry, or differential equations. The broad adoption of the CCSS represents a unique opportunity to shift mathematics instruction not only toward more focused and coherent content standards but also toward engaging students in mathematical practices as they learn that content. This means that all students experience challenging courses and curricula.

Operational Definition of Challenging Courses and Curricula

The operational definition of *Challenging Courses and Curricula* is summarized in the following outline.

1. Big Mathematical Ideas

- Teach for understanding, including the development of conceptual understanding, strategic competence, and procedural fluency.
- Introduce a mathematical idea by posing problems that motivate it.
- Provide a coherent collection of problems organized around a big mathematical idea.
- Provide opportunities for students to use multiple representations of a mathematical idea.

2. Inquiry and Reflection

- Communicate that learning mathematics should be a sense-making process.
- Ask students to investigate problems rather than demonstrating solutions to the students.
- Ask students to justify their thinking.
- Ask students to engage in reflection.
- Encourage diverse ways of thinking.
- Communicate that both accuracy and efficiency are important.

3 Productive Disposition

- Help students develop persistence, resourcefulness and confidence.
- Help students become autonomous learners.
- Provide a safe, respectful learning environment.

4. Communication

- Promote the development of precise mathematical language.
- Value communication by asking students to explain their ideas orally and in writing.
- Value the role of communication in developing intellectual community in the classroom.
- Establish clear expectations for mathematical assignments.

This definition of Challenging Courses and Curricula was developed by a partnership of nine demographically diverse school districts, a large research university, a small liberal arts college, and an educational nonprofit organization, and there was consensus across all levels about the operational definition.* The partnership is not arguing against offering advanced courses, but rather advocating that every course should provide a challenging learning environment. In elaborating on the Equity Principle (National Council of Teachers of Mathematics, 2000), the NCTM states that "all students need access each year they are in school to a coherent, challenging mathematics curriculum." Classroom practice guided by GBMP's definition of Challenging Courses and Curricula in conjunction with the Standards for Mathematical Practice will result in mathematics courses that challenge all students.

* In the process of developing this definition of challenging courses and curricula, GBMP drew on the National Research Council's (NRC) description of the "intertwined strands of proficiency" in *Adding It Up* (NRC, 2011). We also made use of the "teaching for understanding: guiding principles" articulated in the California State Department of Education *Mathematics: Model Curriculum Guide* [CA] as well as other sources (NRC, 2002; NRC, 2000; Weiss & Pasley (2004); Weiss, Pasley, Smith, Banilower, & Heck, 2003; Charles & Lobato, 1998); Polya, 1984; Bowen, 2007; Parker, 1993; and Parker, 1994 (unpublished course materials developed by the Mathematics Education Collaborative)). We also drew on the expertise of the GBMP National Advisory Board, which includes recognized experts in mathematics, education, and assessment.

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The Importance of Context in Presenting Fraction Problems to Help Students Formulate Models and Representations as Solution Strategies

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That middle grades and high school students have difficulties solving fraction problems is a common perception of both preservice and inservice middle grades (6-8) and secondary (7-12) mathematics teachers with whom we work. This perception is well founded. Working with fractions, especially multiplication and division within a problem context or in ratio and proportion situations, is difficult for students at many ages. In summarizing research on rational numbers and proportional reasoning, Lamon (2007) articulates that "fractions, ratios, and proportions arguably hold the distinction of being…the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science…" (p. 629).

Understanding fractions, together with solving problems in context (e.g., word problems) and algebraic understanding, is often identified as an area that critically affects student success in mathematics. Wu (2009) argues that, "Because fractions are students' first serious excursions into abstraction, understanding fractions is the most critical step in understanding rational numbers and in preparing for algebra" (p. 8). Confrey and Maloney (2010) further articulate the broad spectrum of these issues as follows:

> There is perhaps no more important conceptual area in mathematics education than rational number reasoning. The basis of the multiplicative concepts field (Vergnaud 1983, 1996), rational number reasoning underpins algebra, higher mathematical reasoning, and the quantitative competence required in science. Failure to develop robust rational number construct reasoning and skills in elementary and middle school continues to plague American students. Rational number reasoning is complex, and master represents cognitive synthesis—understanding, distinguishing among, modeling, and interweaving a remarkable assortment of distinct yet closely related concepts over many years. (p. 968)

These struggles with understanding fractions are in addition to the broader challenges students often face in connecting relationships in word problems and algebraic equations (Keiran, 2007). Recently, similar struggles have been identified among preservice and inservice elementary, middle, and secondary mathematics teachers when asked to model or provide representations-based solutions to fraction word problems, with many only able to provide solutions that are primarily procedural and in symbolic form (e.g., Sjostrom, Olson, and Olson, 2010; Olson and Olson, 2011). These finding raise questions about the extent to which teachers are prepared to address these challenges with their students.

Although mathematical content is important, the context within which the mathematics content is situated is also critically important. In fact, in the Common Core State Standards for Mathematics (CCSS, 2010), the second *Standard of Mathematical Practice* (SMP 2) addresses the importance of students' abilities to contextualize and decontextualize quantitative relationships as follows:

> Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. (CCSS, p. 6).

Such recognition of the importance of contextualizing mathematics is not new, but the way contextualization occurs has historically been the topic of debate. In our discussions, we focus on the point of view offered by Boaler (1993) (drawing on the work of Lave (1988)) who suggested that, "the specific context within which a mathematical task is situated is capable of determining not only general performance but choice of mathematical procedure" (p. 13). It is primarily through this lens that we identify our notion of the importance of context with respect to the problem presented in this article, namely, a problem in which certain mathematical procedures, or in our case, representations and models, arise as primary solution strategies.

For the past several years we have engaged in examining the work of middle grades students as well as preservice elementary, middle, and high school mathematics teachers relative to how they use modeling and representation methods to solve word problems involving fractions (Slovin, Olson, and Zenigami, 2007; Olson, Zenigami, and Slovin, 2008; Olson, Slovin, and Zenigami, 2009; Sjostrom, Olson, and Olson, 2010; Olson and Olson, 2011). Four word problems have been consistently used in these studies, and data have been collected from approximately 30 students in Grade 5, 120 students in each of Grades 6 – 8, 40 preservice elementary teachers, and 40 preservice and inservice teachers of Grades 7 – 12.

In this paper, we analyze the responses of a few selected students, preservice mathematics teachers, and inservice mathematics teachers to one contextualized fraction problem using a case study approach. We analyze how individuals express their understandings using the context of the problem to mathematically model a solution without immediately resorting to decontextualized algorithms and discuss the suggestions we have offered elementary, middle, and secondary mathematics teachers, as well as teacher and school district leaders, for how to encourage students and teachers to recognize and take advantage of opportunities to more robustly develop conceptual and contextual understandings of fraction concepts.

A Contextualized Fraction Problem

One of the four problems consistently used in our research is the following contextualized problem called the Painting Problem: *It takes 3/4 liter of paint to cover 3/5 m^2. How much paint is needed to paint 1 m^2? Explain your reasoning and justify your answer.*

Before reading responses of students and teachers given in the article, consider the following prompts and questions as you contemplate finding a solution to this problem:

1. What does a solution to this problem look like? What would be considered a model for the problem situation? What is an equation for the problem situation? How much paint is needed for a 1 square meter board, given the parameters of the context?

2. What work or explanations would we expect to see if this problem was being posed to 5th grade students as their first introduction to such a problem and they have not yet been exposed to fraction computational procedures? What explanations (including mathematical models or expressions) should be provided to the students who are struggling or do not understand what their procedure-based solution "means" in the context of the problem?

3. Suppose this problem was posed to 8th or 9th grade students (say in Algebra or Pre-Algebra) who have not yet fully developed the expected facility with algebra. What explanations (including mathematical models or expressions) should be provided to enable the students to follow the logic and mathematics of the problem and associated discussions so that students are reasonably comfortable with the reasoning underlying the solution? That is, what model or representation, different from an algebraic solution, would likely add to students' comprehension of the mathematics of the problem (i.e., ratio and proportion)?

These questions are given to suggest that students who either have not yet been taught procedures for multiplying and dividing fractions, have little experience working with ratio, or have had these experiences and persist in a state of procedural confusion can meaningfully engage in and solve such problems based only on the context of the problem. In context, students' explanations should be based on making sense of the context regardless of the method of solution. The teachers' challenge, then, is understanding how to provide the support needed so students at all levels make sense of their work and reasoning.

The Painting Problem was selected for two reasons: 1) An accurate model or representation of the problem situation almost directly provides the solution; and 2) The two fractions in the problem have the same numerators and one-to-one functional reasoning stemming from these common numerators seems natural in the context of the problem (e.g., 3/4 L covers 3/5 m^2, or 3 of one thing is "mapped" to 3 of another thing).

Selected Responses to the Painting Problem

In what follows, work samples from two 5th grade students and one 8th grade student are examined and discussed. These work samples illustrate several of the successful models and representations we have seen used in solution strategies to the Painting Problem in earlier research efforts (Olson, Zenigami, and Slovin, 2008; Olson, Slovin, and Zenigami, 2009; Slovin, Olson, and Zenigami, 2007). Work samples from three preservice secondary mathematics teachers are then shared. These work samples demonstrate the range of teacher strategies also found in earlier research efforts (Sjostrom, Olson, and Olson, 2010; Olson and Olson, 2011). In particular, these teacher samples show a range that extends from being able to show a solution, to displaying beginnings of a solution strategy but not fully following to a conclusion, to employing a solution or solution strategy similar to that of the students but more elaborate.

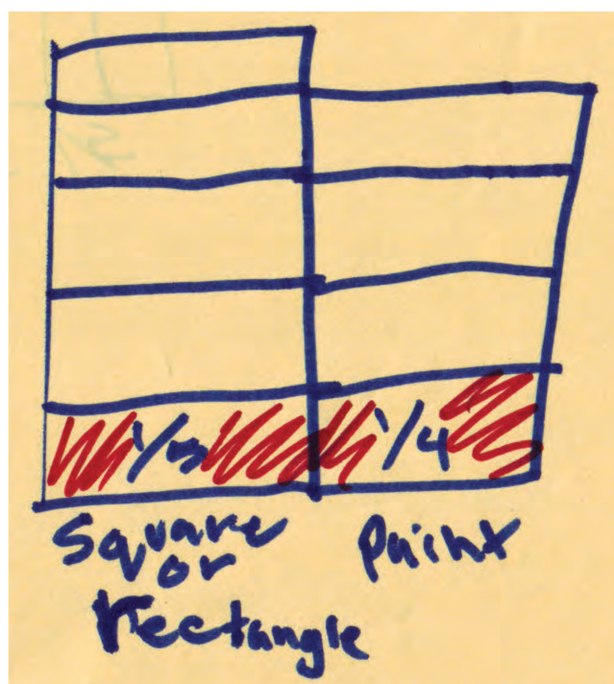
Student Thinking

In Figure 1, Anne[1], a 5th grade student, used area models to represent each quantity (3/4 and 3/5). She drew separate but contiguous area regions to represent each fraction in the problem, and made 1/4 liter and 1/5 square meter the same height, creating unit fraction models derived from the correspondence between 3/4 liter of paint and 3/5 square meter. In her verbal explanation of the model, Anne articulated her use of the one-to-one correspondence between 1/4 liter of paint and 1/5 square meter. Using this correspondence, Anne knew covering 1 square meter (that is, 5/5 square meter) required 5/4 liter of paint.

Anne's original drawing only had the word square listed for the square meter. As she articulated her reasoning, explained her thinking, and justified her conclusion, she felt the need to indicate what she had drawn was a representation of a "square or rectangle." Although she reasoned through the problem using this rectangular representation, she added the words "or rectangle" to the diagram. This reasoning indicated a tension between the object being represented (a *square* meter board) and what was used to represent a square (a rectangular bar).

The impetus for this tension was, again, displayed through her process of justifying her conclusion, and provides evidence of one student's need to verbally align her thinking to her visual model although her visual model is perhaps not a precise representation of the problem situation. These issues of precision in verbal descriptions and visual representations highlight underlying challenges in fostering students thinking with regard to the CCSSM *Standards for*

FIGURE 1: *Anne's Model*



[1] All student names used in this paper are pseudonyms.

Mathematical Practice. In particular, although Anne was not as precise as she could have been with her visual model in representing the problem context, her verbal description eventually did precisely describe her visual model through the process of justifying her answer. Consequently, there are many levels of precision (SMP 6) at play through her process of justifying her conclusion (SMP 3).

Jason, another 5th grade student, used reasoning similar to Anne's but he used different notation (Figure 2). While Jason did not draw a model of the square meter or paint, he verbalized the relationship between 1/4 liter and 1/5 square meter using the correspondence between 3/4 liter and 3/5 square meters, suggesting a mental model of the problem. Jason's thinking led him to the correct solution (5/4 liter) with an appropriate explanation that maintained the one-to-one correspondence until 5/5 square meters (i.e., 1 square meter) was attained. Although Jason's use of the equations 3/4 = 3/5 and 1/4 = 1/5 are not mathematically correct as written, he used these notations as tools to organize his thinking about the one-to-one relationship inherent in the problem. This allowed him to reason through the problem and obtain a correct solution.

Jason's use of imprecise mathematical notation presents another example in which such notation or *symbolic* representation facilitated a student's understanding of the context. This instance once again illustrates the complexity with which teachers will need to approach the implementation of the *Standards for Mathematical Practice*. Precision (SMP 6) is critically important for anyone engaging in mathematical thinking, argumentation, and justification. Additionally, students' emerging visual and symbolic representations must be understood as indicators of their present mental constructs and structures. Consequently, although Jason's symbolic representation proved helpful to him in attaining and justifying a solution, his work also presents an opportunity for his teacher to question Jason to help him reflect on his understanding of the "meaning of the symbols [he chose], including using the equal sign consistently and appropriately" (CCSS, p. 7). That is, Jason's use of the equals sign between 1/4 and 1/5, as well as 3/4 and 3/5 (and 1¼ and 5/5), could potentially be found to be a matter of implementing a "place holder" symbol due to not yet having engaged in discussion and experiences related to ratios and appropriate ratio notations.

Thus, Jason's emerging understandings and mental constructs should not be wholly discounted, nor should the teacher accept at face value his use of the equal sign as implying "equality" simply because it facilitated the accurate answer. Rather, Jason's work must become an opportunity for discussion of the mathematical content, as well as the mathematical practices and representations used to arrive at an answer.

At the time that Anne and Jason were solving this contextualized fraction problem, they had not yet received formal instruction related to division of fractions, ratios, or ratio notation. Yet these students (and others) were able to develop or visualize a model or representation that, in essence, helped them attain a correct solution. That is, creating the model or representation was, itself, a highly

FIGURE 2: *Jason's Model*

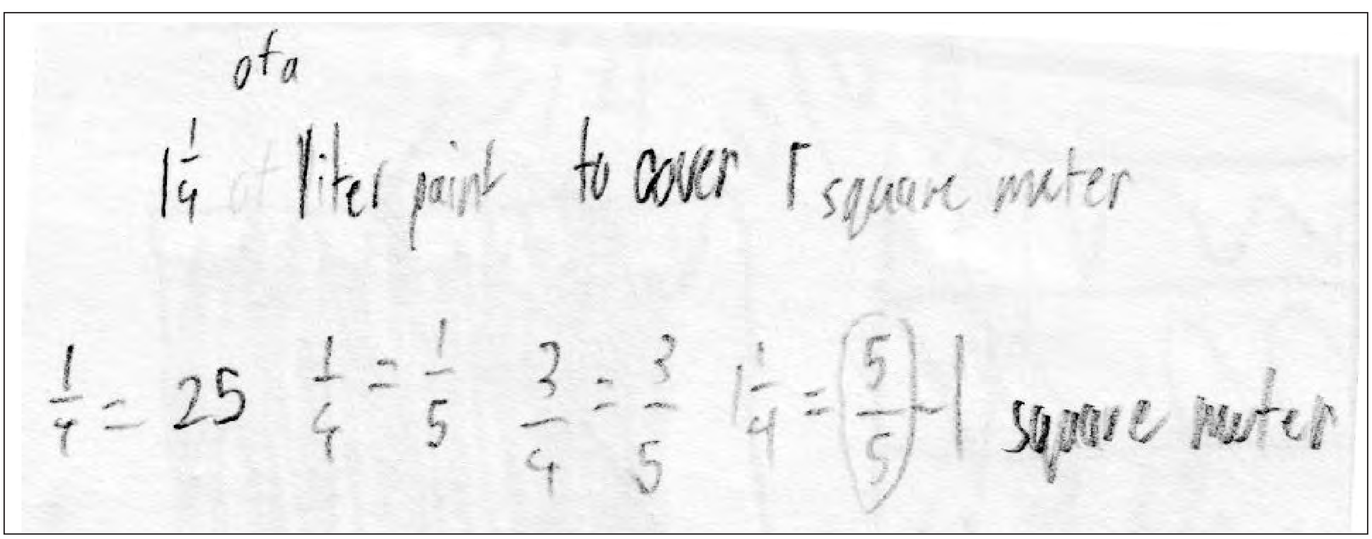
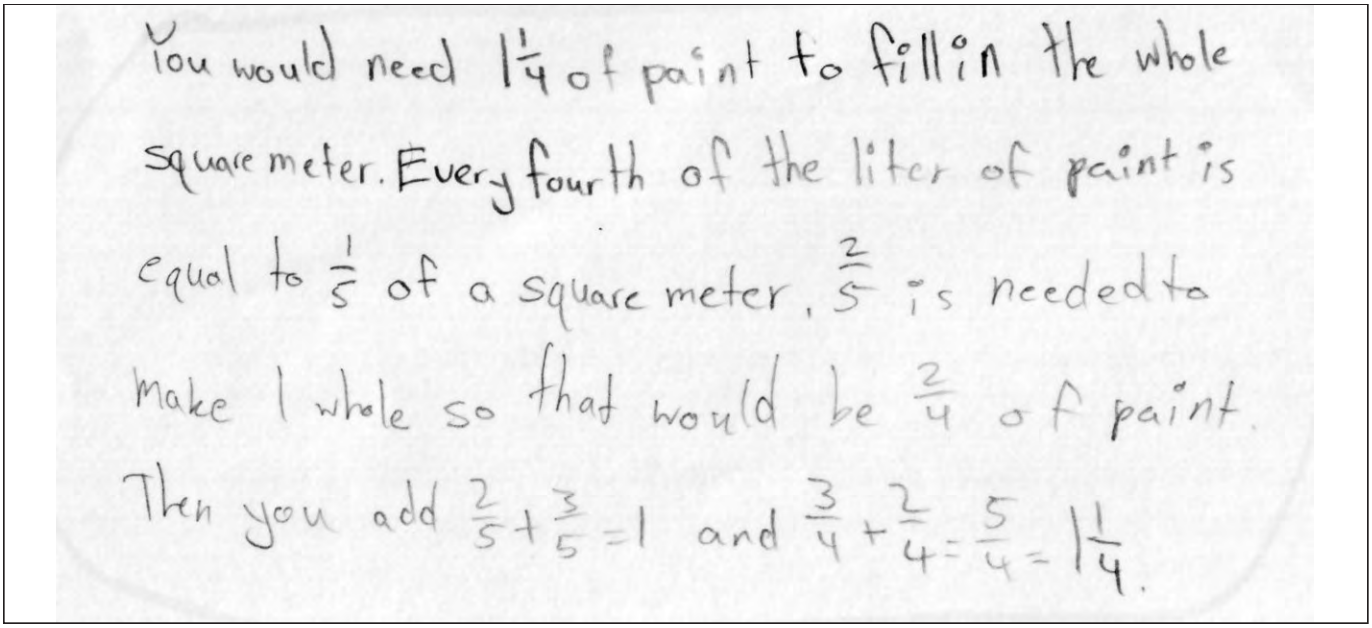


FIGURE 3: *Joseph's Work*


You would need $1\frac{1}{4}$ of paint to fill in the whole square meter. Every fourth of the liter of paint is equal to $\frac{1}{5}$ of a square meter, $\frac{2}{5}$ is needed to make 1 whole so that would be $\frac{2}{4}$ of paint. Then you add $\frac{2}{5}+\frac{3}{5}=1$ and $\frac{3}{4}+\frac{2}{4}=\frac{5}{4}=1\frac{1}{4}$.

effective solution strategy. In Figure 3 the work of a 6th grade student, Joseph, shows that although he is better able to articulate his thinking, the underlying ideas based on the context of the problem situation are similar to that displayed by the 5th grade students Anne and Jason.

These three students provided thoughtful arguments and justifications (SMP 3), looked for and made use of the structure in the problem context (SMP 7), but displayed varying degrees of precision (SMP 6) in their representations and arguments, if one solely examines their written artifacts.

This kind of problem is not a simple exercise for many middle school students despite having received formal instruction addressing ratio and proportion as well as operations with fractions. In fact, more than half of middle school students asked to solve this problem were not able to do so successfully and very few of these students even attempted a visual model or representation in their effort to find a solution (Olson, Slovin, and Zenigami, 2009). These types of problems have also been found to present teachers with difficulties when asked to provide a visual or other descriptive representation that incorporates minimal symbolic mathematics.

Teacher Thinking

Figures 4, 5, and 6 show the responses of three preservice teachers. In Figure 4, the preservice teacher presents a representation of the fractions in the problem but does not successfully use the model to obtain a solution. This preservice teacher correctly identifies that 2/5 of the square meter is yet to be covered, and attempts to use various equations, but to no avail. In Figure 5, the preservice teacher appears to "know" that more than one liter of paint is needed, and identifies a question that would help solve the problem: "You need what fraction of 3/4 liter is needed to complete this painting?" However, this preservice teacher does not use the representation to reason that 2/3 of the 3/4 liter is needed to complete the square meter. How this reasoning would be useful can be seen in the preservice teachers' use of the model where the second "3/4 liter" is used. The second 3/4 liter covers pieces 4, 5, and 6 of the square meter; however, only pieces 4 and 5 need to be covered (i.e., 2/3 of the 3/4 liter). There is appropriate thinking displayed in the work shown in Figure 5, but the preservice teacher was not able to use the model to find the solution.

In Figure 6, the preservice teacher makes use of the representations to solve and explain the solution to the problem. This preservice teacher does not rely on a "procedure" involving symbolic or algebraic manipulations, but rather, provides reasoning in the context of the relationship between 1/4 liter and 1/5 square meter. This preservice teacher showed an understanding of the problem and used a successful strategy to explain relationships rather than attempting a solution via a rule, such as 3/4:3/5 as 1:x. In

FIGURE 4: *Work of a preservice teacher who is not able to solve the problem*

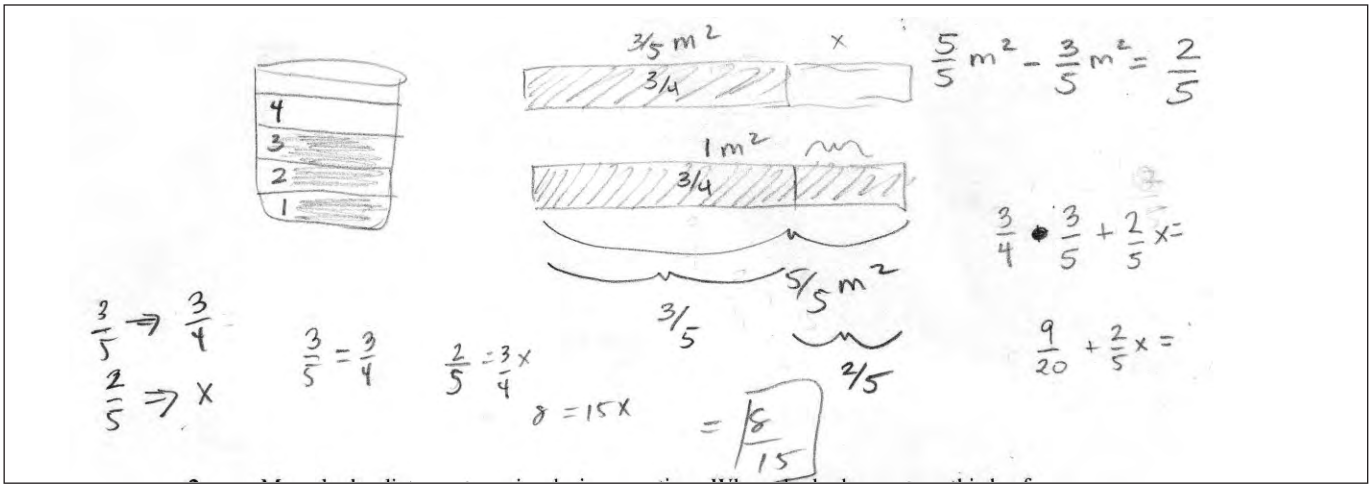


FIGURE 5: *Work of a preservice teacher who employs a model but cannot see how to use it finish solving the problem*

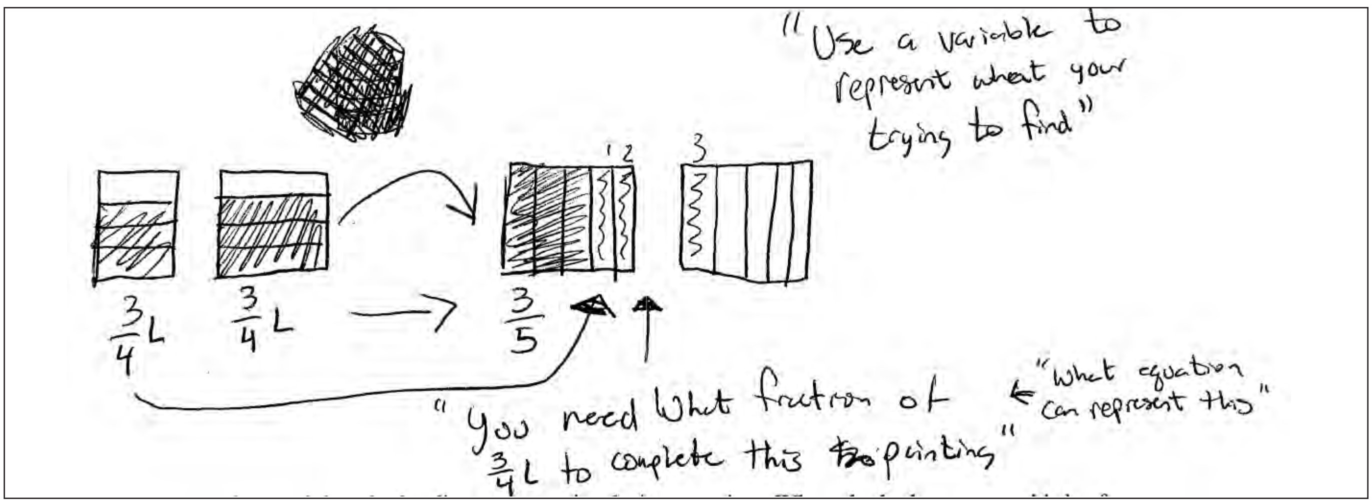
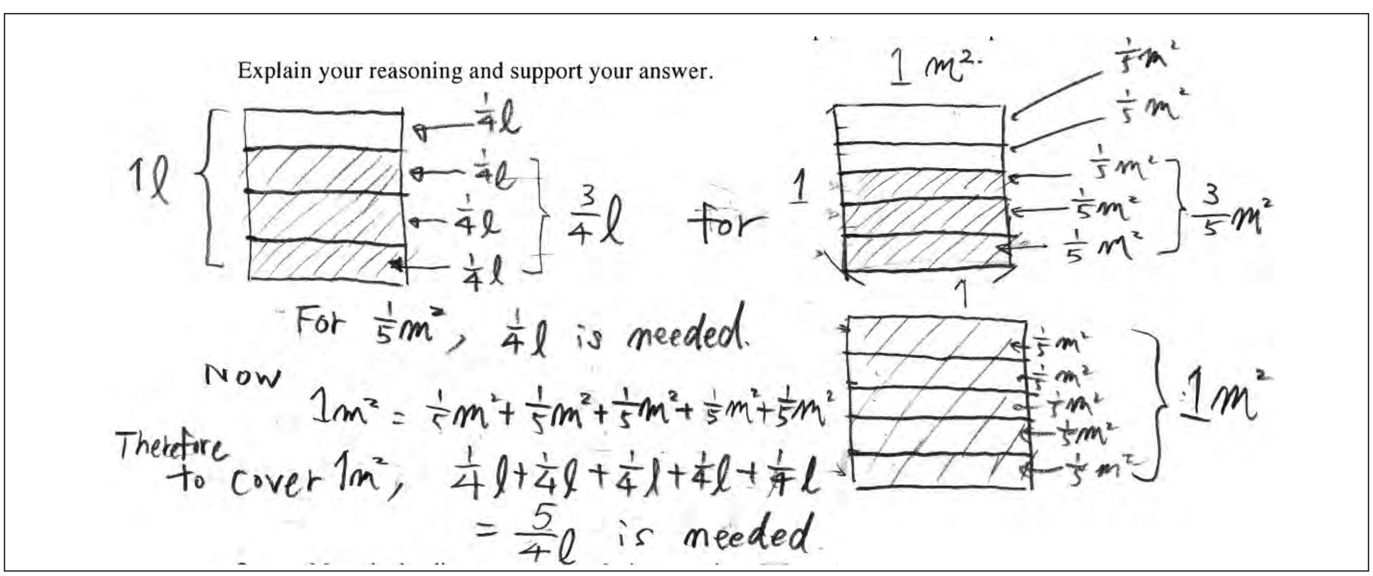


FIGURE 6: *Work of a preservice teacher who uses a model to solve and explain a correct solution*



essence, this teacher's reasoning is similar to that used in the 5th grade students' explanations.

These discussions of teachers' understandings and abilities to represent mathematical contexts are consistent with those of prior research. Such research has shown that teachers appear to not have coherent ideas on how to start thinking about the problem; are able to provide initial thoughts on a solution strategy and use a model up to a point, but do not go further; or are able to provide solution strategies similar to those of the middle school students highlighted in the earlier discussion of student thinking (Sjostrom, Olson, and Olson, 2010; Olson and Olson, 2011).

Commonalities in Student and Teacher Thinking

In the prior research involving responses of approximately 30 students in Grade 5, 120 students in each of Grades 6 – 8, 40 preservice elementary teachers, and 40 preservice and inservice teachers of Grades 7 – 12, when a student or teacher used a visual model or representation to correctly solve the Painting Problem, the model or representation was similar to the examples provided. What was it about the problem or its context that led to the use of a model or representation? Those using a model or representation used a unit rate approach, but the presence of a common numerator may have been instrumental in making that choice, perhaps making visible a one-to-one correspondence between the 1/4 liter of paint and 1/5 square meter. It is our view that the students' use of one-to-one correspondence in the examples shared was fundamentally different from the usual unit rate approach. Importantly, students in Grade 5 and Grade 6 (pre-CCSSM) likely have not encountered ratios in any structural or mathematical sense. Rather, these students (and students like them) move to a "unit numerator" based on the one-to-one correspondence inherent in the problem context – that, if 3 will cover 3, then 1 will cover 1. Thus, the numerators are unitized to make more explicit the one-to-one correspondence, and not as a matter of procedurally unitizing a ratio.

Unfortunately, preservice and inservice teachers often relegate the importance of solution strategies that utilize modeling and representations to the realm of "lesser mathematics." However, the importance of these approaches is well articulated by Wu (2011) in his discussion of a model used to solve a problem involving fractions as follows:

> We see plainly that there is no need to use multiplication of fractions for the solution, and moreover, no need to memorize any solution template. The present method of solution makes the reasoning very clear" (pp. 36-37).

Furthermore, the solutions provided by Anne, Jason, and Joseph exemplify Lamon's (2001) view that, "current instruction in fractions grossly underestimates what children can do without help." (p. 153).

This is not to say that using a model or representation always leads to a correct solution. We saw that some students and preservice teachers were able to create an appropriate beginning model or representation but were unable to finish the problem. A common error involved using common denominators to solve the problem, frequently adding 15/20 (3/4 liter of paint) with 8/20 (the 2/5 square meter left to be painted) to achieve an answer of 23/20. Such solutions suggest a rush to the use of rules and procedures rather than thoughtful use of the context of the problem to find a solution that makes sense.

Conclusion

Campbell, Rowan, and Suarez (1998) argue because algorithms are important, teachers should know and be able to use various strategies for finding a solution, and assist students in making sense of processes and procedures to determine if their work is reasonable. In other words, it is critically important for teachers to "sense-make an algorithm" in various contexts. Through the process of sense making and conceptually understanding algorithms, we argue that teachers' are able to mathematically understand and engage their students' misconceptions. We are not arguing against the importance for teachers and students to be able to symbolically and procedurally arrive at a solution to a word problem involving fractions. However, we suggest that without displaying the ability to understand and use the context of a problem to arrive at a solution through modeling, foundational and conceptual mathematical knowledge is likely not well developed.

It is important to understand how the use of modeling and representations in certain contexts allows for the appropriate conceptual development of key algorithms.

For example, when should the algorithm "flip (invert) and multiply" for division of fractions emerge as contextually making mathematical sense? In Grade 6 standards (i.e., 6.NS.1) when fraction division occurs in a "story context" or by way of "visual fraction models?" Perhaps. Importantly in this CCSSM standard, the context, associated representations, and justification for general (algorithmic) relationships between division and multiplication are all essential components to mathematical sense making.

Additionally, providing a story (or visual) context for which unit rates are computed with respect to division of complex fractions, but only as a solution strategy to this particular contextualized fraction division problem, is arguably a mathematically appropriate context through which teachers can extend students conceptions and misconceptions regarding algorithmic procedures. In such a context, the denominator is inverted and multiplied by the numerator to find a new rate (numerator) per unit (denominator). The CCSSM identifies such a context in Grade 7 (7.RP.1).

Thames and Ball (2010) indicate that, "No one would argue with the claim that teaching mathematics requires mathematics knowledge.…by better understanding the mathematical questions and situations with which teachers must deal, we would gain a better understanding of the mathematics it takes to teach" (p. 221). Furthermore, Keeley and Rose (2006) note that, "Teachers may not be aware of the misconceptions and alternative ideas their students hold, and sometimes, they harbor those very same misconceptions" (p. 6).

There is room for growth in teachers' understandings of the use of context, models, and representations in exploring and solving fraction word problems, particularly with problems that are able to be solved using non-procedural models and representations. Stylianou (2010) indicates that teachers conceptions of representation as a process and practice need further development to include representations more successfully in instruction, especially for non high-performing students. Although there is not uniform agreement on the nature of representations, Stylianou reasons that, "symbolic expressions, drawings, written words, graphical displays, numerals, and diagrams are all representations of mathematical concepts" (p. 326). In this paper, illustrations were provided of how students and teachers both use models or representations effectively as solution strategies.

What is it that we as teachers, teacher leaders, and school leaders can do to help students and teachers reason through problems such as the Painting Problem and use representations or models to assist their thinking? To help students and teachers develop better problem solving abilities related to fractions, we suggest the following.

First, recognize that the development of fraction understanding is a challenge and the way we structure the introduction to the use of fractions to students is very important. Wilson, Edgington, Nguyen, Pescosolido, and Confrey (2011) give an indication of a learning trajectory related to fractions, and indicate that children's early experiences must provide a solid basis for future applications.

Second, recognize the importance of problems in context. As shown in the representations and verbal explanations of Anne and Jason, the words used to situate a problem, if modeled well, provide direction enough so that students can successfully solve the problem. As teachers, the responsibility of including such problems in students' mathematical experiences lies with us. As noted by Sullivan, Zevenbergen, and Mousley (2003) in their discussion of the importance of the context of mathematics tasks, a primary issue in teaching mathematics is that, "teachers need to be fully aware of the purpose and implications of using a particular context at a given time" (p. 111).

Third, reconsider the usefulness of representation and modeling as viable solution strategies. It is important to recognize that students need as much practice in these modeling and representation strategies as they need practice with procedural and algorithmic strategies. The ability to model problem situations and arrive at solutions through the use of those models is not simple or easy to master. With each model or representation used by a student, a teacher needs to practice asking, "How does your model or representation demonstrate what the problem is saying, and how will you use that to help you understand or solve the problem?" Abrahamson (2006) notes the following:

> …one can use these representations without appreciating which ideas they enfold and how these ideas are coordinated. Consequently, learners who, at best, develop procedural fluency with these representations, may not experience a sense of understanding, because they lack opportunities to bridge the embedded ideas, even if these embedded ideas are each familiar and robust. (p. 464)

For this reason, talking about the meaning of the model and represent used in a solution strategy is as important as using it to solve the problem at hand.

In this paper, we shared solutions to a fraction problem and discussed how the context of the problem was useful in finding a solution. We examined the commonalities of the solutions provided by elementary and middle school students and preservice secondary teachers. We discussed how considering the context of the mathematics within problems can be useful for teachers and teacher leaders for helping build situations in which students are asked to model or provide representations for their work. We hope our suggestions are of use to many of you in your mathematics leadership roles.

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Effective Professional Development: Defining the Vital Role of the Master Teacher

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Originally conceptualized in 1987 as a bridge between research mathematicians at Rice University and the precollege mathematics education community, the Rice University School Mathematics Project (RUSMP) has evolved over time, transcending its initial goal, and now serving as a nationally recognized K-12 mathematics education center with a documented ability to improve teacher knowledge and student learning (e.g., Cruz, Turner, & Papakonstantinou, 200; Killion, 2002a, 2002b, 2002c; McCoy, Hill, Sack, Papakonstantinou, & Parr, 2007; Parr, Papakonstantinou, Schweingruber, & Cruz, 2004; Troutman, 2011). RUSMP nurtures and prepares mathematics teachers to become a collaborative community of highly-skilled, K-12 mathematics educators capable of providing effective mathematics instruction to all students regardless of race, gender, socioeconomic status, mathematics aptitude, or prior success in mathematics.

RUSMP's mission is to help teachers and school administrators better understand the nature of mathematics and to provide effective teaching and assessment of mathematics, equipping all students for success as they encounter mathematics in today's society. To achieve this mission, RUSMP is based upon the principle that teachers learn best from fellow teachers, identified as master teachers, who are knowledgeable and experienced. Although RUSMP offers a wide variety of programs and support for the K-12 educational community, the cornerstone of RUSMP is its Summer Campus Program (SCP) initiated through National Science Foundation (NSF) funding (TEI 86-52030 and TEI 9055501). This paper discusses the selection, development, characteristics, roles, and impact of teacher leaders identified by RUSMP as SCP master teachers.

A key to the success of SCP are master teachers, who are K-12 classroom teachers serving as instructors, role models, and mentors for participants in the program. Initially developed in 1987 as a six-week course with a primary focus on developing the content knowledge of 48 middle and high school mathematics teachers, SCP is now a four-week program that serves 80-120 K-12 teachers annually in classes separated into grade bands who participate in a rigorous program that explores all aspects of contemporary mathematics education including mathematics content, instruction, assessment, and issues related to access and equity in the classroom. These grade bands include elementary (K-3), intermediate (4-6), middle (7-8), and high (9-12).

Selection and Development of the RUSMP Master Teacher

In our SCP program, master teachers have always been selected for their abilities as exemplary teachers as identified by Rice University faculty and local district mathematics leaders. During the first three years of SCP, master teachers worked closely with Rice University faculty from the mathematics, mathematics sciences, statistics, and computer science departments to prepare lessons on advanced mathematics topics such as linear algebra, number theory, mathematical induction, and mathematical modeling. It was expected that master teachers would develop a better understanding of advanced mathematics through their interactions with Rice University faculty as they prepared these lessons (Capper, 1987). As school district needs changed, so did SCP. Today, SCP targets teachers identified as most benefiting from intensive long-term professional development in mathematics content and pedagogy, including induction year teachers and teachers who are new to teaching mathematics. Over the years, as the focus of the mathematics content shifted from exploring advanced mathematics to developing a deep understanding of the precollege mathematics that K-12 teachers are expected to teach, the paradigm for selecting and developing master teachers changed. In recent years, SCP focuses on pedagogy as well as precollege mathematics. Rice University faculty no longer selects master teachers; the faculty serves as a resource for current master teachers and makes presentations for SCP participants. New master teachers are now selected by RUSMP directors with recommendations from current master teachers through a process that includes observations of candidates as they interact with both students and with other teachers.

Presently, new master teachers are mentored by current master teachers and RUSMP directors with assistance from the Rice University mathematics faculty rather than solely by Rice University mathematics faculty, as today's master teachers are charged with many more tasks than in the early years. These include helping novice teachers with classroom management and discipline, modeling differentiated instruction and assessment techniques, demonstrating how to organize classrooms for student-centered instruction, leading book studies, and incorporating more technology into instruction (e.g., interactive white boards, web sites).

Characteristics of a RUSMP Master Teacher

The founding directors of RUSMP described RUSMP master teachers as precollege teachers who were "recognized by their peers or administrators in either a formal or informal way as being among the best in the teaching profession, and whose practices it would be good, in principle, for other teachers to emulate" (Austin, Herbert, & Wells, 1990). Killion (2011) defines teacher leaders as:

> …teachers who have both more experience and a level of expertise as a professional educator not typical in novice teachers. This perspective of teacher leadership acknowledges that one grows into a leadership role through a wide range of experiences and formal and informal professional development. (p. 7)

One of the founding directors of RUSMP, the current director of RUSMP (one of the first RUSMP master teachers), and former and current RUSMP master teachers completed questionnaires and participated in focus group discussions designed to create a profile of the characteristics of master teachers. The characteristics that emerged mirror the combined perspectives of Austin, Herbert, and Wells (1990) and Killion (2011). RUSMP master teachers today are expected to do the following:

- act as role models whose practices would be good for other teachers to emulate;
- utilize interpersonal skills to connect with teacher participants;
- motivate teacher participants to learn content and pedagogical skills;
- develop curriculum for SCP;
- share their passion and enthusiasm with others about the content being taught;
- demonstrate through presentations, publications, and lesson modeling;
- facilitate professional learning opportunities; and
- realize the importance of their professional growth.

In addition, RUSMP master teachers acknowledge that they have grown into their leadership roles and are recognized by their peers and administrators as reflective leaders who are among the best in the profession, with knowledge and skills to affect change. The RUSMP master teachers come together regularly as a professional learning community to discuss issues related to pedagogy and policy and to further their own personal professional development. They also participate in RUSMP professional development sessions (e.g., book studies, technology implementation, assessment techniques). In addition, master teachers continue to grow further through their participation in professional organizations and by attending and presenting at conferences.

Tasks of the SCP Master Teacher

Using the information from the questionnaires and focus group discussions, a job analysis for the position of SCP master teacher was conducted. Three broad task categories emerged as being fundamental to the job of a master teacher:

- developing curriculum materials and resources for SCP;
- determining individual characteristics and abilities of participants; and
- presenting lessons incorporating both mathematical content and recommended pedagogical practices.

Each of these roles is discussed further, below.

Developing curriculum materials and resources for SCP.

> When you teach the right things the right way, motivation takes care of itself. If students aren't enjoying learning, something is wrong with your curriculum and instruction – you have somehow turned an inherently enjoyable activity into drudgery. (Brophy, 1998, p. 1)

Master teachers motivate participants to learn through developing curriculum materials that include academic activities that are engaging, meaningful, and worthwhile. Master teachers possess considerable knowledge of current practices in education including the National Council of Teachers of Mathematics' (NCTM) mathematical content standards of number and operations, algebra, geometry, measurement, and data analysis and probability and the process standards of problem solving, reasoning and proof, communication, representation, and connections (NCTM, 2000). These mathematical content and process standards have had longstanding importance in K-12 mathematics education. These same NCTM standards as well as the strands of mathematical proficiency specified in the National Research Council's (NRC) report *Adding It Up* (NRC, 2001) are embedded in the Standards for Mathematical Practice described in the Common Core State Standards (Common Core State Standards Initiative, 2011). As curriculum specialists, master teachers utilize their wealth of knowledge of the mathematical processes and proficiencies within the standards of NCTM and the Common Core to develop the participants' essential understandings of mathematical content and pedagogical skills.

Creativity is required, as master teachers incorporate a variety of everyday materials that connect the real world and mathematics, such as newspapers, menus, cereal boxes, cans, tennis balls, hula hoops, string, coffee filters, and measuring spoons. Master teachers strive to empower teachers with an increased understanding of mathematics by promoting the investigation of mathematical concepts in the real world and by linking the mathematics learned in the classroom to mathematics encountered outside the classroom (Troutman, 2011).

Master teachers select appropriate resources for their classes that can be used to illustrate the mathematical concept being explored. This requires considerable knowledge of the various classroom manipulatives that are available, as well as knowledge of how the manipulatives can be most effectively incorporated into the lesson. During lesson preparation, careful consideration is given to the effective use and integration of technology such as calculators, computers, interactive white boards, tablets, online environments to support collaboration and course management, and web-based instruction in the classroom. Lesson preparation also includes incorporating age-appropriate children's literature, field trips, guest speakers, articles, journals, and resource books.

Further, master teachers possess ample organizational and planning skills and the ability to work collaboratively with others. As there are two master teachers in each class with RUSMP directors serving as advisors in planning the instructional process, master teachers never work in isolation; instead they are able to capitalize on the strengths of other professionals and university professors. The importance of this collaboration cannot be emphasized enough, whether it is among master teachers or teachers, as it promotes professional growth. The National Council of Supervisors of Mathematics (NCSM) has identified the theme of teacher collaboration and professional learning as essential when contemplating the specific domains of leadership focus and responsibility (NCSM, 2012).

Determining Individual Characteristics and Abilities of Participants. Another major category of tasks of master teachers focuses on gauging the initial ability levels of participants and how best to assist them in the learning process. Master teachers move about the classroom listening to group discussions and providing input as requested to clarify questions the participants may have. Consequently, well-developed observational and effective listening skills are essential, allowing master teachers to assist participants who are having difficulty, particularly in a group setting. This process occurs even if participants are unaware that they do not fully understand the concept or are unwilling to acknowledge that they have lack of understanding. As master teachers interact with participants one-on-one as well as in group settings, they gauge their participants' individual comfort level with the material and utilize a certain level of sociability and interpersonal skills to effectively listen to participants and respond with sensitivity to their needs. In addition, master teachers use pre- and post-surveys as well as participants' reflections in daily journals to gauge participants' ability levels and growth.

Fundamental gauges of participants' ability levels are assessments that provide information on teachers' mathematics and pedagogical knowledge (see Appendix for sample assessment items). Master teachers create these assessments and administer them to participants at both the beginning and the end of SCP to assess knowledge and pedagogical growth. A certain degree of creativity in developing unique and thought-provoking mathematical content questions, along with strong pedagogical content knowledge, is required to develop meaningful assessments.

Formative assessments are utilized as a vehicle for master teachers to provide ongoing feedback to participants through their daily journals. Technology is incorporated as participants enter these journal writings into a class management system. Participants become accustomed to this personalized feedback of providing suggestions and pedagogical tips to enhance their specific instructional practices and needs.

Presenting Lessons Incorporating Both Mathematical Content Information and Recommended Pedagogical Practices. The final, and most crucial, aspect of the master teacher's job is the presentation of lessons encompassing both the mathematical content and the pedagogical skills needed to effectively convey that knowledge. Therefore, master teachers must be able to speak accurately and fluently about complex topics before a class of teachers and be able to interact comfortably with a second instructor. In addition, master teachers must possess an extensive knowledge of mathematics content in the grade level they are addressing and beyond that grade level, including an understanding of nationally accepted standards for teaching that mathematics content.

Technology and manipulatives are used to better illustrate the concepts being explored in SCP. The seamless integration of technology and manipulatives into instruction demands that master teachers have knowledge of multimedia technology, social networking, instructional apps, interactive white boards, tablets, educational software, graphing calculators, data collection devices, and internet sites, as well as knowledge of the capabilities and limitations of such technology.

Master Teachers as Role Models

One significant overarching role of master teachers is their responsibility for serving as role models for other teachers. Throughout SCP, master teachers provide participants in the program with implicit examples of developing and teaching lessons, involving students in discussions, and working with other educators in the planning and implementation of effective lessons. These opportunities help participants develop an understanding of pedagogical content knowledge that might be used by participants to successfully engage students.

Over time, master teachers have recognized that role modeling builds the self–efficacy of participants. Self-efficacy has been found to have a direct positive relationship with performance (Bandura, 1997; Bandura & Locke, 2003; Tschannen-Moran & McMaster, 2009), and higher self-efficacy can lead to setting more challenging goals (Bandura, 1997; Williams, T. & Williams, 2010), which are associated with higher performance (Locke & Latham, 1990). Social psychology has found that role modeling can enhance a person's self-efficacy, or confidence in one's own abilities (Bandura, 1986, 1997).

SCP participants, by observing master teachers, can develop a thorough understanding of the complexity of the tasks these master teachers are performing, and they can detect how to best manage aspects of the tasks that might arise in unexpected situations (Gist & Mitchell, 1992; Moberg, 2000). These observations help instill within participants the idea that if the master teachers can do it, they can, too.

Recent Results

Evidence from the 2010 and 2011 years of RUSMP's SCP indicates that its master teacher methodology does indeed improve participants' self-efficacy. Each year, participants are administered questionnaires at both the beginning and at the end of their experience with SCP. Questions address their beliefs about teaching and learning mathematics, their evaluation of the program itself, and their feelings of preparedness in the following seven instructional areas:

- Presenting the applications of mathematical concepts
- Using cooperative learning groups
- Considering students' prior conceptions about mathematics when planning curriculum and instruction
- Using hands-on activities to introduce and develop math concepts
- Managing a class of students who are using manipulatives
- Using technology as an integral part of math instruction
- Using a variety of methods to assess students' mathematical knowledge

Pretest and posttest survey data were collected from all 84 participants in the 2010 SCP and 76 of the 80 participants in the 2011 SCP and results indicate that participants felt more confident in their ability to teach mathematics following their completion of the program.

Upon completion of the program, most participants reported feeling fairly well prepared or very well prepared in presenting the applications of mathematical concepts (97.6% in 2010 and 94.8% in 2011 as shown in Table 1), using cooperative learning groups (98.8% in 2010 and 97.4% in 2011 as shown in Table 2), taking into account students' prior conceptions about mathematics when planning curriculum and instruction (95.2% in 2010 and 98.6% in 2011 as shown in Table 3), using hands-on activities to introduce and develop math concepts (100% in 2010 and 100% in 2011 as shown in Table 4), managing a class of students who are using manipulatives (100% in 2010 and 100% in 2011 as shown in Table 5), using technology as an integral part of math instruction (90.4% in 2010 and 96.1% in 2011 as shown in Table 6), and using a variety of methods to assess students' mathematical knowledge (97.7% in 2010 and 98.6% in 2011 as shown in Table 7).

Paired samples t-tests performed on aggregated data for all classes indicated that participants' sense of preparedness had increased significantly ($p < .05$) in all instructional areas except for using a variety of methods to assess students' mathematical knowledge over the course of the program in 2010. Disaggregated data revealed very similar results for the participants in the elementary and intermediate classes focused on grade level bands, K-3 and 4-6, respectively. Participants in the middle and high school grade level classes also showed gains in their sense of preparedness in the same six instructional areas. However, participants in these upper grade level classes showed gains that were statistically significant for three or four of the instructional areas.

In 2011, paired samples t-tests performed on aggregated data for all grade level bands of participants indicated that their sense of preparedness in all of the seven instructional areas had increased significantly ($p < .001$) over the course of the program. Comparable results were apparent for all or for six of the seven instructional areas for all grade level bands in 2011 ($p < .05$). Means and standard deviations of these ratings are presented in Table 8 for the 2010 SCP and in Table 9 for the 2011 SCP.

Table 1: Results for RUSMP SCP 2010 and 2011 participants' post-program self-ratings on "After your experience, how well prepared do you feel you are to present the applications of mathematical concepts?"

	Results from RUSMP SCP 2010		Results from RUSMP SCP 2011	
	Frequency	Percent	Frequency	Percent
Not well prepared	0	0	0	0
Somewhat prepared	2	2.4	4	5.3
Fairly well prepared	23	27.4	17	22.4
Very well prepared	59	70.2	55	72.4
Total	**84**	**100.0**	**76**	**100.0**

Table 2: Results for RUSMP SCP 2010 and 2011 participants' post-program self-ratings on "After your experience, how well prepared do you feel you are to use cooperative learning groups?"

	Results from RUSMP SCP 2010		**Results from RUSMP SCP 2011**	
	Frequency	Percent	Frequency	Percent
Not well prepared	0	0	0	0
Somewhat prepared	1	1.2	2	2.6
Fairly well prepared	23	27.4	17	22.4
Very well prepared	60	71.4	57	75.0
Total	**84**	**100.0**	**76**	**100.0**

Table 3: Results for RUSMP SCP 2010 and 2011 participants' post-program self-ratings on "After your experience, how well prepared do you feel you are to consider students' prior conceptions about mathematics when planning curriculum and instruction?"

	Results from RUSMP SCP 2010		**Results from RUSMP SCP 2011**	
	Frequency	Percent	Frequency	Percent
Not well prepared	0	0	0	0
Somewhat prepared	4	4.8	1	1.3
Fairly well prepared	19	22.6	22	28.9
Very well prepared	61	72.6	53	69.7
Total	**84**	**100.0**	**76**	**100.0**

Table 4: Results for RUSMP SCP 2010 and 2011 participants' post-program self-ratings on "After your experience, how well prepared do you feel you are to use hands-on activities to introduce and develop math concepts?"

	Results from RUSMP SCP 2010		**Results from RUSMP SCP 2011**	
	Frequency	Percent	Frequency	Percent
Not well prepared	0	0	0	0
Somewhat prepared	0	0	0	0
Fairly well prepared	12	14.5	14	18.4
Very well prepared	71	85.5	62	81.6
Total	**83**	**100.0**	**76**	**100.0**

Table 5: Results for RUSMP SCP 2010 and 2011 participants' post-program self-ratings on "After your experience, how well prepared do you feel you are to manage a class of students who are using manipulatives?"

	Results from RUSMP SCP 2010		**Results from RUSMP SCP 2011**	
	Frequency	Percent	Frequency	Percent
Not well prepared	0	0	0	0
Somewhat prepared	0	0	0	0
Fairly well prepared	21	25.0	20	26.3
Very well prepared	63	75.0	56	73.7
Total	**84**	**100.0**	**76**	**100.0**

Table 6: Results for RUSMP SCP 2010 and 2011 participants' post-program self-ratings on "After your experience, how well prepared do you feel you are to use technology as an integral part of math instruction?"

	Results from RUSMP SCP 2010		**Results from RUSMP SCP 2011**	
	Frequency	Percent	Frequency	Percent
Not well prepared	0	0	0	0
Somewhat prepared	8	9.6	3	3.9
Fairly well prepared	40	48.2	31	40.8
Very well prepared	35	42.2	42	55.3
Total	**83**	**100.0**	**76**	**100.0**

Table 7: Results for RUSMP SCP 2010 and 2011 participants' post-program self-ratings on "After your experience, how well prepared do you feel you are to use a variety of methods to assess students' mathematical knowledge?"

	Results from RUSMP SCP 2010		**Results from RUSMP SCP 2011**	
	Frequency	Percent	Frequency	Percent
Not well prepared	0	0	0	0
Somewhat prepared	2	2.4	1	1.3
Fairly well prepared	25	29.8	22	28.9
Very well prepared	57	67.9	53	69.7
Total	**83**	**100.0**	**76**	**100.0**

Table 8: Paired t-test results for RUSMP SCP 2010 participants' pre- and post-program self-ratings of preparedness for instruction in each category

	All Classes N=84		Elementary (K-3) N=19		Intermediate (4-6) N=23		Middle School (7-8) N=21		High School (9-12) N=21	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Present the applications of mathematical concepts	3.23 (.73)	3.68 (.52)	3.26 (.65)	3.84 (.38)	3.22 (.85)	3.83 (.39)	3.10 (.77)	3.43 (.60)	3.33 (.66)	3.62 (.59)
	$t(83) = -4.69$, $p = .000$		$t(18) = -3.64$, $p = .002$		$t(22) = -3.48$, $p = .002$		$t(20) = -1.44$, $p = .167$		$t(20) = -1.45$, $p = .162$	
Use cooperative learning groups	3.15 (.81)	3.70 (.49)	3.32 (.58)	3.95 (.23)	3.04 (1.00)	3.78 (.42)	3.19 (.81)	3.48 (.51)	3.10 (.77)	3.62 (.59)
	$t(83) = -6.42$, $p = .000$		$t(18) = -4.61$, $p = .000$		$t(22) = -4.10$, $p = .000$		$t(20) = -2.03$, $p = .055$		$t(20) = -2.59$, $p = .018$	
Consider students' prior conceptions about mathematics when planning curriculum and instruction	3.01 (.74)	3.68 (.56)	3.37 (.60)	3.95 (.23)	3.04 (.83)	3.74 (.54)	2.86 (.57)	3.38 (.67)	2.81 (.81)	3.67 (.58)
	$t(83) = -8.52$, $p \leq .000$		$t(18) = -4.16$, $p = .001$		$t(22) = -5.25$, $p = .000$		$t(20) = -2.95$, $p = .008$		$t(20) = -4.95$, $p = .000$	
Use hands-on activities to introduce and develop math concepts	3.04 (.84)	3.85 (.36)	3.37 (.69)	4.00 (0)	3.13 (.82)	3.91 (.29)	2.86 (.86)	3.76 (.44)	2.79 (.92)	3.74 (.45)
	$t(81) = -9.04$, $p = .000$		$t(18) = -4.03$, $p = .001$		$t(22) = -4.41$, $p = .000$		$t(20) = -4.66$, $p = .000$		$t(20) = -4.87$, $p = .000$	
Manage a class of students who are using manipulatives	3.23 (.78)	3.75 (.44)	3.63 (.50)	4.00 (0)	3.26 (.69)	3.74 (.45)	3.14 (.85)	3.62 (.50)	2.90 (.89)	3.67 (.48)
	$t(83) = -6.67$, $p = .000$		$t(18) = -3.24$, $p = .005$		$t(22) = -3.45$, $p = .002$		$t(20) = -2.91$, $p = .009$		$t(20) = -3.93$, $p = .001$	
Use technology as an integral part of math instruction	3.05 (.87)	3.32 (.65)	3.00 (.91)	3.44 (.71)	3.04 (.77)	3.43 (.66)	3.00 (.78)	3.05 (.59)	3.15 (1.09)	3.35 (.59)
	$t(81) = -2.65$, $p = .010$		$t(17) = -1.92$, $p = .072$		$t(22) = -2.40$, $p = .025$		$t(20) = -.326$, $p = .748$		$t(19) = -.748$, $p = .464$	
Use a variety of methods to assess students' mathematical knowledge	3.99 (.11)	3.65 (.53)	4.00 (0)	4.00 (0)	4.00 (0)	3.61 (.50)	4.00 (0)	3.43 (.51)	3.95 (.22)	3.62 (.67)
	$t(83) = 5.85$, $p = .001$				$t(22) = 3.76$, $p = .001$		$t(20) = 5.16$, $p = .000$		$t(20) = 2.32$, $p = .031$	

Note: Standard deviations in parentheses; Mean on a 4-point scale (Anchored with 1 = not well prepared and 4 = very well prepared).

Table 9: Paired t-test results for RUSMP SCP 2011 participants' pre- and post-program self-ratings of preparedness for instruction in each category

	All Classes N=76		Elementary (K-3) N=18		Intermediate (4-6) N=19		Middle School (7-8) N=19		High School (9-12) N=20	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Present the applications of mathematical concepts	3.04 (.70)	3.67 (.58)	2.94 (.64)	3.78 (.43)	2.84 (.83)	3.79 (.54)	3.11 (.66)	3.53 (.61)	3.25 (.64)	3.60 (.68)
	$t(75) = -6.07, p = .000$		$t(17) = -5.72, p = .000$		$t(18) = -3.51, p = .003$		$t(18) = -2.65, p = .016$		$t(19) = -1.68, p = .110$	
Use cooperative learning groups	3.13 (.62)	3.72 (.51)	3.11 (.68)	3.94 (.24)	3.21 (.54)	3.68 (.48)	3.05 (.78)	3.68 (.58)	3.15 (.49)	3.60 (.60)
	$t(75) = -7.41, p = .000$		$t(17) = -5.72, p = .000$		$t(18) = -2.96, p = .008$		$t(18) = -3.31, p = .004$		$t(19) = -3.33, p = .004$	
Consider students' prior conceptions about mathematics when planning curriculum and instruction	2.96 (.74)	3.68 (.50)	3.06 (.73)	3.83 (.38)	2.79 (.86)	3.74 (.45)	3.05 (.71)	3.63 (.50)	2.95 (.69)	3.55 (.61)
	$t(75) = -7.79, p = .000$		$t(17) = -5.10, p = .000$		$t(18) = -4.53, p = .000$		$t(18) = -3.28, p = .004$		$t(19) = -3.04, p = .007$	
Use hands-on activities to introduce and develop math concepts	2.99 (.83)	3.82 (.39)	3.39 (.70)	4.00 (00)	2.79 (.86)	3.74 (45)	3.00 (.94)	3.74 (.45)	2.80 (.70)	3.80 (.41)
	$t(75) = -8.31, p = .000$		$t(17) = -3.72, p = .002$		$t(18) = -4.26, p = .000$		$t(18) = -3.24, p = .005$		$t(19) = -5.63, p = .000$	
Manage a class of students who are using manipulatives	3.18 (.76)	3.74 (.44)	3.44 (.62)	3.72 (.46)	2.89 (.88)	3.79 (.42)	3.32 (.67)	3.68 (.48)	3.10 (.79)	3.75 (.44)
	$t(75) = -6.10, p = .000$		$t(17) = -2.05, p = .056$		$t(18) = -4.46, p = .000$		$t(18) = -2.11, p = .049$		$t(19) = -3.58, p = .002$	
Use technology as an integral part of math instruction	2.89 (.87)	3.51 (.58)	3.00 (.84)	3.83 (.38)	2.58 (1.02)	3.42 (.61)	2.89 (.81)	3.16 (.60)	3.10 (.79)	3.65 (.49)
	$t(75) = -5.93, p = .000$		$t(17) = -4.12, p = .001$		$t(18) = -3.44, p = .003$		$t(18) = -1.76, p = .096$		$t(19) = -2.60, p = .017$	
Use a variety of methods to assess students' mathematical knowledge	2.86 (.71)	3.68 (.50)	2.83 (.71)	3.72 (.46)	2.68 (.89)	3.63 (.50)	2.95 (.71)	3.74 (.56)	2.95 (.51)	3.65 (.49)
	$t(75) = -9.80, p = .000$		$t(17) = -6.47, p = .000$		$t(18) = -4.26, p = .000$		$t(18) = -4.83, p = .000$		$t(19) = -4.77, p = .000$	

Note: Standard deviations in parentheses; Mean on a 4-point scale (Anchored with 1 = not well prepared and 4 = very well prepared).

Conclusion

> The most promising forms of professional development engage teachers in the pursuit of genuine questions, problems, and curiosities, over time, in ways that leave a mark on perspectives, policy, and practice. They communicate a view of teachers not only as classroom experts, but also as productive and responsible members of a broader professional community. (Little, 1993, p. 131)

The professional development provided by RUSMP master teachers is more than just short-term, traditional, instructor-focused mathematical content delivery. Through the leadership and role modeling exhibited by RUSMP master teachers, participants gain content knowledge and develop informally as teacher leaders. RUSMP master teachers are more than mentors; they serve as sources of information and as collegial peers who help guide their fellow teachers.

Overwhelmingly positive changes in participants' self-ratings of their preparedness and self-efficacy for mathematics instruction demonstrated powerful consequences for participants involved in these leadership and role modeling relationships. At the conclusion of the program, participants in all classes reported greater levels of preparedness to present applications for mathematical concepts, use cooperative learning groups, consider students' prior conceptions about mathematics when planning curriculum and instruction, use hands-on activities to introduce and develop math concepts, manage a class of students who are using manipulatives, and use technology as an integral part of math instruction.

In the current climate of high-stakes testing, student assessment, in particular, is a highly charged topic. Mindful of the associated responsibilities and pressures often experienced by teachers, master teachers exposed participants to innovative and emerging techniques of formative and summative assessments, strategies for implementing them, as well as practical ways for utilizing assessment results to improve teaching and learning. However, the only instructional area in which participants' mean self-rating of preparedness did not improve was on their use of a variety of methods to assess students' mathematical knowledge. It is highly possible that RUSMP's comprehensive and integrated approach to student assessment may have provided participants with a quite different frame of reference between the pre- and post-program survey administration regarding this area of instruction. Therefore, it is likely that evidence of the program's effectiveness in the particular instructional area may have been masked by response shift bias (Howard, 1980).

Participants also have achieved high levels of self-efficacy through the leadership and role modeling provided by the RUSMP master teachers. Peer relationships can provide an individual with information about career strategies, performance feedback, and friendship and emotional support beyond what a traditional, hierarchical, mentor/mentee relationship can offer (Kram & Isabella, 1985; Parker, Hall, & Kram, 2008). Clearly, both participants and their schools can benefit from such a professional development paradigm.

This paper investigated the selection, development, characteristics, roles, and impact of teacher leaders identified by RUSMP as SCP master teachers. The roles of the master teachers have evolved over time as SCP has evolved to include a wide variety of educational leadership positions such as curriculum specialists, mentors, role models, motivators, resource providers, and experts in the field of mathematics. These roles are vital to effective teacher professional development.

Appendix

Sample Assessment Items

1. How does learning mathematics from a measurement perspective influence a child's understanding of numeric relationships? Give an example of a measurement activity for your grade level to justify your response.

2. The diagonals in a quadrilateral are perpendicular to each other and bisect the vertex angles of the quadrilateral. Circle all of the figures below that always have these properties.

I Rectangle

II Square

III Rhombus

IV Parallelogram

V Kite

VI Isosceles Trapezoid

3. SAT math scores are scaled so that they are approximately normal, with the mean about 511 and the standard deviation about 112. A college wants to send letters to students scoring in the top 20% on the exam. What SAT math score should the college use as the dividing line between those who get letters and those who do not?

4. Select an algebraic concept, and then describe how you could use manipulatives AND computer technology to teach this concept. Then explain the geometric connection shown by using one or the other. Give an example of one such algebra problem, and draw a graphic representation.

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