NCSM Journal
of Mathematics Education Leadership

FALL/WINTER 2012-2013 VOL. 14, NO. 2

National Council of Supervisors of Mathematics www.mathedleadership.org

Table of Contents

Challenging Courses and Curricula: A Model for All Students

Bernadette Mullins, *Birmingham-Southern College*
Patty Lofgren and Ruth Parker, *Mathematics Education Collaborative*
Barry Spieler, *Montgomery College*
Faye Clark, Rachel Cochran, Ann Dominick, Jason Fulmore, John Mayer, and Sherry Parrish,
University of Alabama at Birmingham

Introduction

The Common Core State Standards (CCSS) have been adopted by 45 states and the District of Columbia as of May 2012. The Standards for Mathematical Content, which describe the content to be taught at each grade level, have received much attention by state boards of education, school districts, administrators and teachers. The Standards for Mathematical Practice describe "varieties of expertise that mathematics educators at all levels should seek to develop in their students," including sense-making, reasoning, perseverance, and communicating mathematical arguments, and while these standards are also vitally important, they have received less attention.

Teachers and administrators in the Greater Birmingham Mathematics Partnership (GBMP)[1] believe that the Standards for Mathematical Practice have received less attention because: (1) teachers and administrators do not understand what some of the mathematical practices are trying to describe; (2) many teachers were taught in traditional lecture style and have never experienced learning in an environment focused on developing the mathematical practices (Mayer, Cochran, Mullins, Dominick, Clark, & Fulmore, 2011); (3) teachers struggle to envision what classrooms would look like where students learn content through engaging in the Standards for Mathematical Practice; and (4) many administrators and teachers focus on the Standards for Mathematics Content as the way to raise test scores and see the Standards for Mathematical Practice as less essential.

Early in the GBMP project, partners collaborated to define "Challenging Courses and Curricula" and this definition has shaped professional development model that, for the past seven years, has promoted classroom instruction consistent with the Standards for Mathematical Practice across the K-12 grade levels and at the undergraduate and graduate levels as well.

When teachers and administrators refer to "challenging" mathematics courses, they are often referring to only the most advanced coursework available (such as a calculus course taken in high school) or to an accelerated track of courses (such as an algebra course taken in 7th grade). A different conception of challenging courses was developed by GBMP with the support from the National Science Foundation Math Science Partnership program and also appears in the literature (US Department of Education, 2008; US Department of Education, 1997). GBMP's definition for challenging mathematics courses asserts that all courses can and should be challenging for the students who take them and should result in students who develop expertise with the Standards for Mathematical Practice. In this article we define challenging courses and provide examples of classroom practice guided by this definition.

[1] Award #EHR-0632522

The GBMP project believes that challenging courses and curricula (1) help students deepen their knowledge of the big ideas in mathematics; (2) promote student inquiry and reflection; (3) support the development of productive disposition; and (4) foster articulate written and oral communication. We also recognize that aligned assessment practices positively impact these four overarching goals.

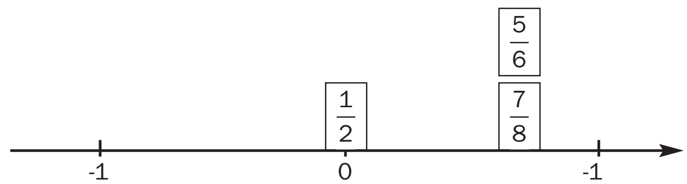
In our project, we are seeing classrooms where students are highly engaged in solving complex mathematics tasks, where students make sense of the mathematics they are doing, and where "talking mathematics" is the norm. All students are engaged but no student is held back from taking the mathematics as far as possible. In these classrooms, teachers think of mathematics as a sense-making discipline and help students make connections between and among seemingly unrelated mathematical ideas rather than viewing mathematics as sets of isolated skills and domains. What we see is consistent with our definition of challenging courses and curricula. We describe below the classroom environment and instructional practices found in these contexts.

Classroom Environment and Instructional Practices in Challenging Courses

1. BIG MATHEMATICAL IDEAS

In challenging courses, students investigate a coherent collection of problems organized around big mathematical ideas. Rather than focusing on isolated skills on an accelerated timeline, challenging courses focus on going deeply into the mathematical study of a few big ideas. In short, we fully appreciate the seemingly contradictory notion that by teaching fewer mathematics topics, but teaching them more thoroughly, learners will come to understand more mathematics and understand it as a fabric of connected and related ideas. This is consistent with the CCSS that emphasize learning critical content in depth.

In a challenging course, a whole class problem might be used to launch an investigation of some of the big mathematical ideas of fractions such as comparing and ordering, defining the whole, equivalence, and magnitude. These problems are often selected based on their potential to build understanding and reveal misconceptions. On one visit to a challenging classroom, we observed the teacher starting with a number line from -1 to +1 on the board. Students were asked to discuss with partners how they would order the following fractions: 1/2, 1/4, 7/8, 5/6, 2/4, 1/3, 1/5, and 3/5. After the discussion, each partner group placed one fraction on the number line. After all fractions were placed, the teacher asked students to discuss whether they agreed with the placement of the fractions and why. During the ensuing whole group discussion, the following big ideas and misconceptions emerged.



One student put 1/2 at 0 with the justification that 1/2 is halfway between -1 and +1. Another student said he thought 1/2 should be placed between 0 and 1 because 1 is the whole, and 1/2 is half of the whole, like half of a candy bar. In response to these ideas, and to focus students' attention on defining the whole, the teacher asked if 1/2 could be placed at both places.

Two partner groups argued that 7/8 and 5/6 were equivalent and should be at the same place on the number line because they were both one part away from the whole. Other students disagreed because 1/8 is smaller than 1/6 and so 7/8 is closer to one.

Throughout this lesson, students were developing Standards for Mathematical Practice (MP) including the following:

- Make sense of problems and persevere in solving them (MP 1);
- Reason abstractly and quantitatively (MP 2);
- Construct viable arguments and critique the reasoning of others (MP 3); and
- Attend to precision (MP 6).

2. INQUIRY AND REFLECTION

GBMP's conception of challenging courses is based on the belief that coming to know and understand important mathematical ideas takes time and that learning occurs through a process of inquiry and reflection. We view confusion—the cognitive dissonance that accompanies "not knowing"—as a natural and even desirable part of the process of constructing new knowledge. Challenging courses provide opportunities for students to struggle with problems, to find their own ways of solving them, and to recognize that there is usually not just one way to solve a problem. The dilemma for teachers is that they were often taught that a teacher's job is to teach how to best solve problems by giving clear explanations of each step to take in the solution. We have learned, however, that this natural inclination to want to put confusion to rest, and to "help" those who are struggling, is often counterproductive when it comes to developing mathematical understandings and productive dispositions.

We want to clarify our use of the word "confusion" and not leave the impression that we view all confusion as desirable. Some kinds of confusion need to be cleared up, especially when "social knowledge" is involved. For example, the use of a symbol may need to be explained or the language used in posing a problem may warrant clarification. But we have come to believe that teaching by telling rarely leads to deep mathematical understandings or productive mathematical dispositions. When students ask for help, teachers interact with them in ways that do not direct their thinking, listening to their thinking and asking probing questions in order to help students find their own ways through the problems.

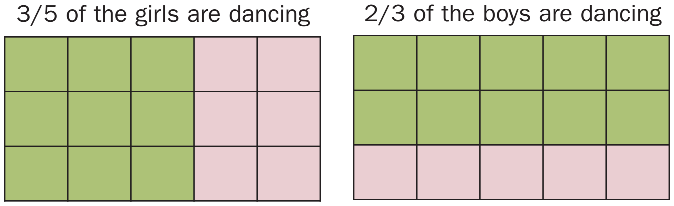
To illustrate, we describe an observation of students in a middle school classroom investigating the following Square Dance problem:

> For the first dance at the school square dance, 2/3 of the boys danced with 3/5 of the girls. What fraction of the students were dancing?

We recommend that you stop and think about this problem before reading on.

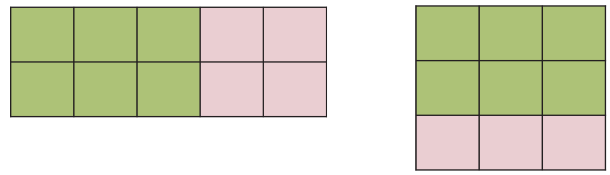
Students worked in small groups using color tiles to represent and make sense of the problem. Initially, one group thought they had a solution, but it involved finding a common denominator. They confronted the confusion that 9/15 + 10/15 = 19/15 is more than 100% of the students. Another group created the following diagram and said that 3/5 of the girls are dancing with 2/3 of the boys so 19/30 of the students are dancing.

3/5 of the girls are dancing   2/3 of the boys are dancing



They confronted the confusion that one dancing boy did not have a partner. Eventually these groups wrestled their way out of their confusion and found a geometric solution that made sense to them. Using the diagram below, they argued that 3/5 of the girls were dancing with 2/3 of the boys, so 12/19 of all the students were dancing.

3/5 of the girls are dancing   2/3 of the boys are dancing



Another group attacked the problem algebraically and reasoned that $\frac{3}{5}G = \frac{2}{3}B$ where ***G*** is the number of girls and ***B*** is the number of boys. This group faced confusion about what to do next and made several unsuccessful attempts, eventually reasoning their way to the following solution that made sense to them. Since $\frac{3}{5}G = \frac{2}{3}B$, the number of boys is 9/10 times the number of girls, $B = \frac{9}{10}G$. Therefore, the fraction of students dancing is:

$$\frac{\frac{3}{5}G + \frac{2}{3}B}{G + B} = \frac{\frac{3}{5}G + \frac{3}{5}G}{G + \frac{9}{10}G} = \frac{\frac{6}{5}G}{\frac{19}{10}G} = \frac{12}{19}.$$

In solving this problem, students were modeling with mathematics (MP 4) in addition to addressing MP 1, 2, 3, and 6.

3. PRODUCTIVE DISPOSITION

Challenging courses are designed with the understanding that learning mathematics involves hard work. Even students who are confident in their mathematical content knowledge often encounter disequilibrium when they are asked to see problems in multiple ways or to solve a problem where the solution path is not immediately obvious to them. All students, no matter their level of competence or confidence, are engaged with mathematical problems that demand perseverance. Students learn what it means to struggle and to experience the satisfaction of finally solving a problem or understanding a mathematical idea. Students come to know that the degree of satisfaction or exhilaration they experience in solving a problem is often directly proportional to the amount of struggle and effort expended.

Challenging courses foster a productive and supportive learning community. Students come to care about each other's learning. They learn that in trying to understand the thinking of others they understand mathematics at a deeper level themselves. They learn how to ask for help by seeking guidance but not answers and they learn how to help other students without doing the mathematical thinking for them. Rather than rescuing students, teachers interact with students in ways that build more powerful mathematical understandings and dispositions that diminish the need for future rescue. Their goal is to help students become autonomous learners.

As an example, we visited a third grade classroom in which students were exploring whether halving and doubling was a strategy that would always work for multiplication. Students had noticed that to find the answer for a multiplication problem, you could halve one factor and double the other factor, and it would still give the same product (e.g., $5 \times 18 = 10 \times 9 = 90$). One group of students discussed that this strategy was good for working with even numbers, but it wouldn't work with two odd numbers. Another student said that if the strategy was going to work, it would have to work in all cases, so let's see if it works with 7×7. The teacher heard this group discussion and knew that this would be a messy problem, but instead of stopping the students or suggesting an easier problem, she encouraged them to give it a try.

Alethia: $7 \times 7 = 49$; double 7 to get 14, and what's half of 7?

Mark: You can halve 6 to get 3 and half of 1 is ½, so half of 7 is 3½.

Shandra: So how do we multiply 3 ½ × 14?

Alethia: $3 \times 14 = 42$, and half of 14 is 7, and $42 + 7 = 49$.

Students: It works! Let's see if we can do it again!

Undaunted, the students proceeded to investigate the problem by halving 3 ½ and doubling 14 (1¾ × 28). The students reasoned their way through this by computing $1 \times 28 = 28$; ½ × 28 = 14; ¼ × 28 = 7, and $28 + 14 + 7 = 49$, which led to cheers and applause at their own effort. The point of this example is not that this group of students figured out how to multiply a mixed number by a fraction (which is not a third grade standard), but that students were exploring properties of multiplication (which is a third grade standard) in an environment that encouraged them to ask their own questions and to persevere in finding the answers. The teacher also understood it was important to ask this group two questions: (1) Will this strategy always work? and (2) When would this be an efficient strategy?

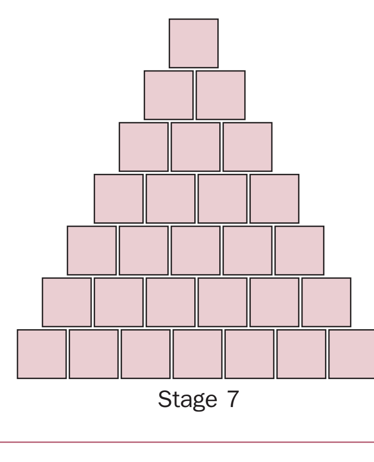
This vignette illustrates that the teacher valued investigation of mathematical ideas and believed students were capable of solving difficult mathematical problems. These 3rd graders believed that mathematics is supposed to make sense and they persisted in their sense-making process. They knew from experience that rich mathematical problems rarely have instant answers and so they were willing to persevere in reasoning through a challenging and unfamiliar problem. While this discussion provides opportunities to develop numerous Standards for Mathematical Practice, it particularly addresses perseverance in solving problems (MP 1) and looking for and making use of structure (MP 7).

4. COMMUNICATION

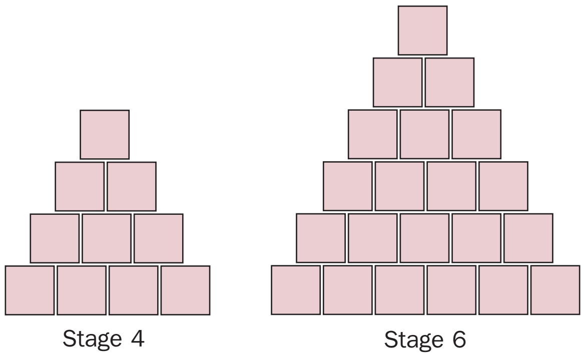
Talking and writing mathematics is the norm in challenging courses. Communication of mathematical thinking occurs in small groups as students work together to make sense of problems and during whole class processing of their thinking. An essential element of whole class processing is establishing a safe environment in which all students and mathematical ideas are treated with respect. During processing, students volunteer to share their diverse ways of seeing and solving problems. As different solutions and various representations (geometric, verbal, numerical, and algebraic) emerge, students deepen their understanding by making connections among various representations and solution paths. Whole class processing is done with an eye toward clarifying the mathematics involved and learning to consider, value, question, and build upon each others' mathematical ideas.

To illustrate, we describe an observation in an algebra class processing the following Building problem:

> A few stages of an increasing pattern are shown below. How many tiles would it take to build Stage 10? What about any stage? (Richardson, 1984).



Stage 7

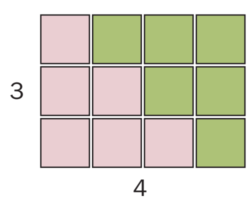


Again, you might want to stop and think about this problem before reading on.

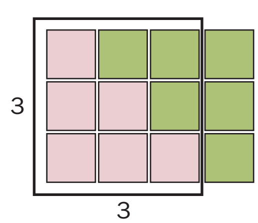
The teacher asked for volunteers and Patricia's hand went up.

Patricia: I was building stage 3, moving tiles around, and I realized I could "left justify" stage 3 to look like this (the diagram on the right below).

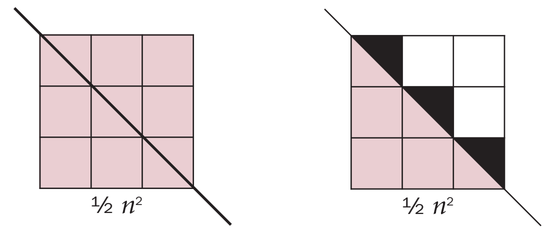
Then I put two copies of stage 3 together like this [see below]. Now it's easy to count that there are 3×4 tiles in all, but that's twice as many as I wanted, so there's really only (3×4)/2 tiles in stage three. For stage n there would be $[n\times(n+1)]/2$ tiles.



Xavier chimed in that he built the same arrangement of tiles as Patricia, but he saw a 3×3 square plus 3 more tiles. Then he also divided by 2. For stage n, his formula was $\frac{p^2 + p}{2}$.



Next JaMichal volunteered that he solved the problem by completing a square with color tiles, dividing the square in half, and adding back half of each tile on the diagonal for a result of $\frac{1}{2} n^2 + n/2$.



In addition to making sense of problems and communicating their ideas to others, students in this class exhibited MP 8 (Look for and express regularity in repeated reasoning). These students investigated the pattern for small values of n until they were able to determine a general formula. Mathematical Practice 5 (Use appropriate tools strategically) was also in evidence here. In this case, the tools in use were manipulatives, but in another problem the tool might be a protractor or a graphing calculator).

Conclusion

This article describes a broadly applicable vision for challenging mathematics courses. Whereas the common interpretation of "challenging" mathematics is relevant only for a small population of students enrolled in accelerated classes or enrichment programs, this definition applies to all mathematics courses and all students. The universality of the definition was one aim of the design—it is applicable not only to the K-12 classrooms described in our examples and to undergraduate and graduate courses and professional development institutes—but is also universal in another sense. Using this definition of challenging courses helps students develop mathematical practices that transcend any particular mathematics course. It builds their capacity to learn as much as it builds their knowledge of arithmetic, or geometry, or differential equations. The broad adoption of the CCSS represents a unique opportunity to shift mathematics instruction not only toward more focused and coherent content standards but also toward engaging students in mathematical practices as they learn that content. This means that all students experience challenging courses and curricula.

Operational Definition of Challenging Courses and Curricula

The operational definition of *Challenging Courses and Curricula* is summarized in the following outline.

1. Big Mathematical Ideas

- Teach for understanding, including the development of conceptual understanding, strategic competence, and procedural fluency.
- Introduce a mathematical idea by posing problems that motivate it.
- Provide a coherent collection of problems organized around a big mathematical idea.
- Provide opportunities for students to use multiple representations of a mathematical idea.

2. Inquiry and Reflection

- Communicate that learning mathematics should be a sense-making process.
- Ask students to investigate problems rather than demonstrating solutions to the students.
- Ask students to justify their thinking.
- Ask students to engage in reflection.
- Encourage diverse ways of thinking.
- Communicate that both accuracy and efficiency are important.

3 Productive Disposition

- Help students develop persistence, resourcefulness and confidence.
- Help students become autonomous learners.
- Provide a safe, respectful learning environment.

4. Communication

- Promote the development of precise mathematical language.
- Value communication by asking students to explain their ideas orally and in writing.
- Value the role of communication in developing intellectual community in the classroom.
- Establish clear expectations for mathematical assignments.

This definition of Challenging Courses and Curricula was developed by a partnership of nine demographically diverse school districts, a large research university, a small liberal arts college, and an educational nonprofit organization, and there was consensus across all levels about the operational definition.* The partnership is not arguing against offering advanced courses, but rather advocating that every course should provide a challenging learning environment. In elaborating on the Equity Principle (National Council of Teachers of Mathematics, 2000), the NCTM states that "all students need access each year they are in school to a coherent, challenging mathematics curriculum." Classroom practice guided by GBMP's definition of Challenging Courses and Curricula in conjunction with the Standards for Mathematical Practice will result in mathematics courses that challenge all students.

* In the process of developing this definition of challenging courses and curricula, GBMP drew on the National Research Council's (NRC) description of the "intertwined strands of proficiency" in *Adding It Up* (NRC, 2011). We also made use of the "teaching for understanding: guiding principles" articulated in the California State Department of Education *Mathematics: Model Curriculum Guide* [CA] as well as other sources (NRC, 2002; NRC, 2000; Weiss & Pasley (2004); Weiss, Pasley, Smith, Banilower, & Heck, 2003; Charles & Lobato, 1998); Polya, 1984; Bowen, 2007; Parker, 1993; and Parker, 1994 (unpublished course materials developed by the Mathematics Education Collaborative)). We also drew on the expertise of the GBMP National Advisory Board, which includes recognized experts in mathematics, education, and assessment.

References

Bowen, E.R. (2007). Student engagement and its relation to quality work design: A review of the literature. *Action Research Exchange*, 2(1).

Common Core State Standards Initiative: preparing America's students for college and career. (2012). Common Core State Standards for Mathematics. Retrieved October 29, 2012, from www.corestandards.org/the-standards

California Department of Education. (1987). Teaching for understanding, in Mathematics: Model Curriculum Guide, Kindergarten through Grade Eight. Sacramento, CA: California State Department of Education.

Charles, R. & Lobato, J. (1998). *Future Basics: Developing Numerical Power.* Reston, VA: National Council of Teachers in Mathematics.

Mayer, J., Cochran, R., Mullins, B., Dominick, A., Clark, F., & Fulmore, J. (2011) Perspectives on Deepening Teachers' Mathematics Content Knowledge: The Case of the Greater Birmingham Mathematics Partnership. In E. M. Gordon, D. J. Heck, K. A. Malzahn, J. D. Pasley, & I. R. Weiss (Eds.), *Deepening teachers' mathematics and science content knowledge: Lessons from NSF Math and Science Partnerships.*

National Council of Teachers of Mathematics. (2000). Principles and Standards for School Mathematics. Reston, VA: National Council of Teachers of Mathematics.

National Research Council. (2002). *Learning and Understanding: Improving Advanced Study of Mathematics and Science in U.S. High Schools.* J.P. Gollub, M.W. Bertenthal, J.B. Labov, & J.C. Curtis, (Eds.).Washington, DC: National Academies Press.

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, and B.Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academies Press.

National Research Council. (2000). J.D. Bransford, A.L. Brown, & R.R. Cocking, (Eds.). *How People Learn: Brain, Mind, Experience, and School: Expanded Edition.* Washington, DC: National Academies Press.

Parker, R. (1993). *Mathematical Power: Lessons from a Classroom.* Portsmouth, NH: Heinemann Press.

Parker, R. (1994). *Working Toward Mathematical Power.* Unpublished manuscript.

Polya, G. (2004). *How to Solve It: A New Aspect of Mathematical Method.* Princeton, NJ: Princeton University Press.

Richardson, K. (1984). Developing Number Concepts Using Unifix Cubes. Lebanon, IN: Addison Wesley.

U.S. Department of Education. (2008). *Foundations for Success: The Final Report of the National Mathematics Advisory Panel.* Retrieved October 29, 2012 from www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf

U.S. Department of Education. (1997). Mathematics Equals Opportunity, White Paper prepared for U.S. Secretary of Education Richard W. Riley. Retrieved October 29, 2012 from http://www.ed.gov.pubs/math/mathemat.pdf

Weiss, I.R. & Pasley, J.D. (2004). What is high-quality instruction?, *Educational Leadership* 61(5), 24-28.

Weiss, I.R., Pasley, J.D., Smith, P.S., Banilower, E.R., & Heck, D.J. (2003). Looking Inside the Classroom: A Study of K-12 Mathematics and Science Education in the United States. Retrieved October 30, 2012, from http://www.horizon-research.com/insidetheclassroom/reports/looking/complete.pdf