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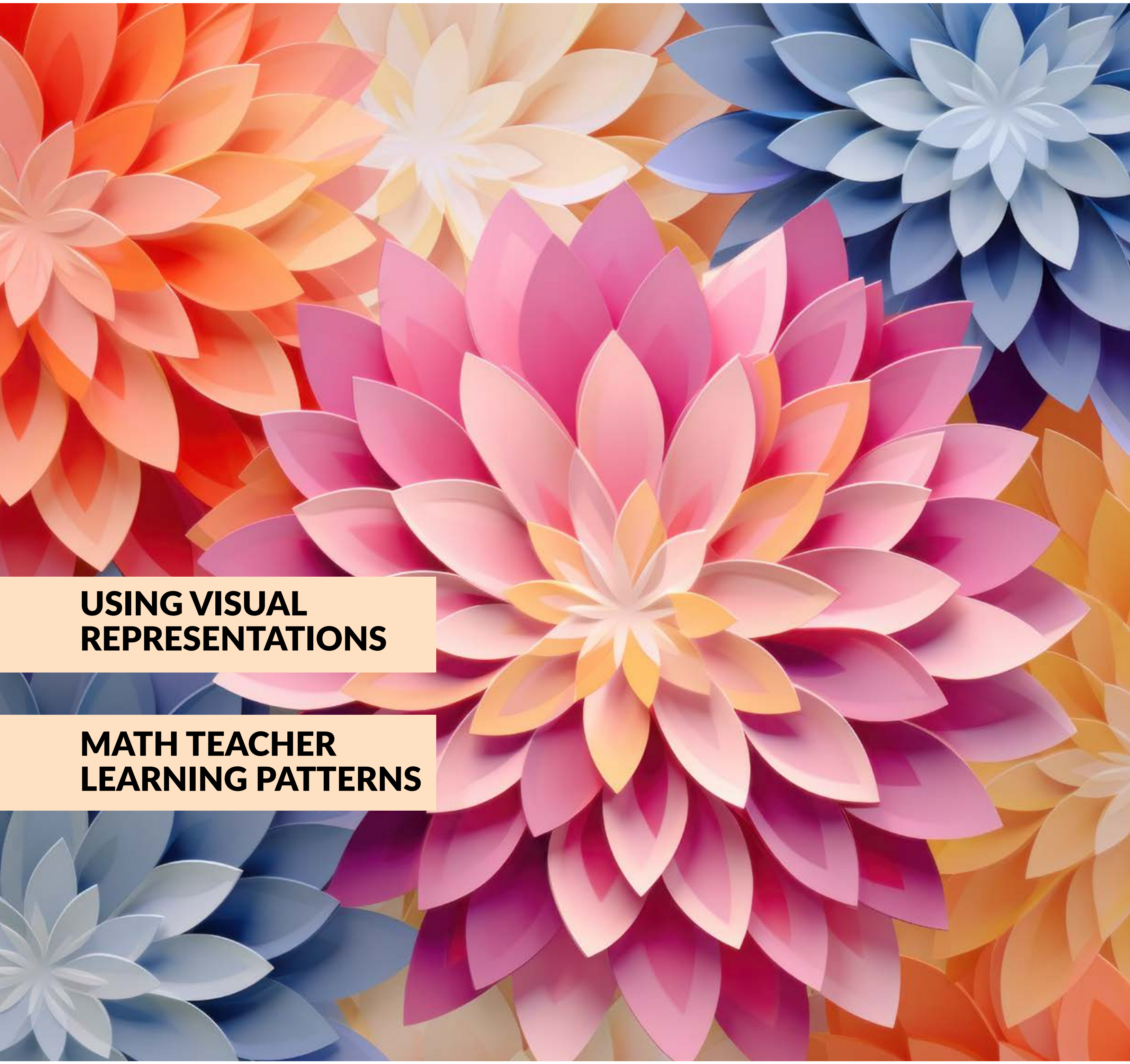

USING VISUAL REPRESENTATIONS

MATH TEACHER LEARNING PATTERNS

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The editors of the ***NCSM Journal of Mathematics Education Leadership (JMEL)*** are interested in manuscripts addressing issues of leadership in mathematics education which are aligned with the NCSM Vision.

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About NCSM

NCMS Vision

NCSM is the premiere mathematics education leadership organization. Our bold leadership in the mathematics education community develops vision, ensures support, and guarantees that all students engage in equitable, high quality mathematical experiences that lead to powerful, flexible uses of mathematical understanding to affect their lives and to improve the world.

NCMS Mission Statement

The purpose of the NCSM Journal of Mathematics Education Leadership is to advance the mission and vision of NCSM by:

• Strengthening mathematics education leadership through the dissemination of knowledge related to research, issues, trends, programs, policy, and practice in mathematics education
• Fostering inquiry into key challenges of mathematics education leadership
• Raising awareness about key challenges of mathematics education leadership in order to influence research, programs, policy, and practice.

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COMMENTS FROM THE EDITORS

Paula M. Jakopovic
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University of Cincinnati

Fall is a time of year that brings with it both beautiful autumn days and hints of the winter yet to come. During this ephemeral season, we say farewell to summer and welcome new beginnings. At the National Council of Supervisors of Mathematics (NCSM), we are also in a time of transition, as the end of our annual conference in October brought with it new leadership, as well as changes for the *Journal of Mathematics Education Leadership (JMEL)*.

This fall, we welcome our new NCSM President, Dr. Katey Arrington, who shared with us her vision for the organization's future in the latest *Insider* issue. In her inaugural remarks, she calls on mathematics education leaders to commit to "making change for the better," which requires leaders to raise the expectations for all partners to better serve all students, as well as to boldly and fearlessly lead the charge in making a positive impact on mathematics education at a systemic level. *JMEL* seeks to take up this call and lead for change by curating manuscripts for our issues that showcase current research and innovative initiatives. Ultimately, our hope is to provide our readers with the necessary tools and resources to navigate the challenges and changes ahead.

JMEL is also excited to announce several changes of our own. First, we welcome our new co-editor, Dr. Evthokia Stephanie Saclarides, to the team. She will serve a two-year term on the board, as co-editor in 2023-2024 with a transition to lead editor in 2024-2025. Additionally, our readers may have noticed that the journal has undergone its own transformation, with a refreshed look for the fall 2023 issue.

In this issue, both articles highlight the potential impact of teacher professional development (PD) models on teachers' use of ambitious mathematics teaching practices (Lampert & Graziani, 2009; Lampert et al., 2010; Schoenfeld, 2023). The first article, "Using Visual Representations: How Using Visual Representations May Provide Teacher Leaders with a Tool for Supporting Sustained Teacher Learning," explores the impact of two different PD models that integrated visual representations as a focal component of teachers' training. In this study, Placa, Koellner, and Seago investigate the long term effects of the PD model on teachers' ability to take up and sustain ambitious teaching practices four years after participating in the PD sessions. The authors share their findings as well as implications for other leaders engaged in teacher PD around the impact of integrating visual representations as a central facet.

The second article, "Math Teacher Learning Patterns: Characterizing Mathematics Teacher Learning Patterns Through Collegial Conversations in a Community of Practice," presents findings from a community of practice PD model that utilized video case studies to engage secondary (6-12) mathematics teachers in their consideration of ambitious instructional materials. DiNapoli, Daniel, Leonard, Kim, Bonaccorso, and Murray illustrate their use of the Teaching for Robust Understanding (TRU) Framework (Schoenfeld & the TRU Project,2016) and interrogation of the data using frame analysis to examine how the PD model shifted participants' conversations in their communities of practice from congenial to collegial via intentional intervention design. The teams provide actionable facilitation practices that other leaders could adopt and adapt to help teachers take up ambitious teaching through collaborative and collegial work.

Both articles offer food for thought on the ways we, as bold leaders, can continue to hone our work providing high-quality, high-impact PD for the teachers in our communities. We hope that the articles inspire you to actively lead the way to re-envision what is possible and implement effective collaborative support in the pursuit of ambitious mathematics teaching at all levels.

REFERENCES

Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In M. K. Stein & L. Kucan (Eds.), *Instructional explanations in the discipline* (pp. 129-141). Springer.

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. *The Elementary School Journal, 109*(5), 491-509. https://doi.org/10.1086/596998

Schoenfeld, A. H. (2023). A theory of teaching. In A.K. Praetorius & C. Y. Charalambous (Eds.), *Theorizing teaching: Current status and open issues* (pp. 159–187). Springer.

Schoenfeld, A. H., & the Teaching for Robust Understanding Project. (2016). *An introduction to the Teaching for Robust Understanding (TRU) framework.* Graduate School of Education. http://map.mathshell.org/trumath.php

USING VISUAL REPRESENTATIONS

HOW USING VISUAL REPRESENTATIONS MAY PROVIDE TEACHER LEADERS WITH A TOOL FOR SUPPORTING SUSTAINED TEACHER LEARNING

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ABSTRACT

This paper highlights two teachers that participated in two different professional development (PD) experiences who sustained new teaching practices and learning four to five years after participating. Both PD projects focused on visual representations (VRs) and encouraged and modeled ambitious teaching practices. Teachers provided video clips and participated in interviews to illustrate and describe changes that took place in their learning and practice. Our qualitative analysis showed that (1) the teachers' use of VRs appears to be strongly connected to teachers' own active learning of VRs in PD, (2) VRs appears to be a key factor that supported the teachers' use of other ambitious teaching practices in their classroom, and (3) that the two teachers remembered and continued to use ambitious practices and VRs in their classrooms in ways that not only aligned to the goals and intention of the PD, but also adapted and extended representations to different mathematical domains and settings. Implications for mathematics education leaders suggest that a focus on VRs may be one tool to anchor learning to deepen teachers' abilities to engage in ambitious teaching practices.

Keywords: professional development, mathematics education, teacher education, professional learning, representations.

HOW USING VISUAL REPRESENTATIONS MAY PROVIDE TEACHER LEADERS WITH A TOOL FOR SUPPORTING SUSTAINED TEACHER LEARNING

Understanding what teachers take up and use from professional development (PD) years after their participation is of great interest to those who lead and study PD. One central challenge for the field is how to design interventions that target teacher knowledge, while also maintaining a focus on instructional practice and student learning (Jacobs et al., 2020). Researchers have worked to address this challenge and there is now a strong research base delineating critical design aspects of effective PD (Borko et al., 2010; Darling-Hammond et al., 2017; Desimone et al., 2002; Heck et al., 2019; Hill et al., 2013). Effective PD contains some agreed upon qualities: a focus on subject matter content, teacher's active learning, collective participation, coherence, and adequate duration (Desimone & Garet, 2015; Garet et al., 2001; Putnam & Borko, 2000). PD also needs to be connected to practice and enable participants to develop their pedagogical content knowledge and implement new strategies in their settings (Ball & Bass, 2003; Ball & Even, 2009; Kennedy, 2016). However, these are necessary but not sufficient conditions as studies of PD outcomes yield a mixed picture. Although some PD programs that adhere to design recommendations by the literature have produced encouraging results (Franke et al., 2001; Kutaka et al., 2017; Taylor et al., 2017), others have proven much less successful (Jacob et al., 2017; Santagata et al., 2010).

These mixed empirical results have led to the call for more research to better understand how teacher PD translates into effective practice (Desimone, 2009; Hill & Papay, 2022; Kennedy, 2016). One hypothesis that could account for these varying results is that most studies about the effects of PD in mathematics education focus primarily on the period immediately after the program activities. Relatively few studies have been conducted on their longer-term effectiveness (Brendefur et al., 2013; Cai & Hwang, 2021; Franke et al., 2001; Zehetmeier & Krainer, 2011). In her review of the literature on the impact of PD, Kennedy (2016) concluded, "the ultimate effects of PD are likely not completely visible at the end of the program year" (p. 960). As a result, it is important to investigate what practices are sustained over time as well as factors that may contribute to long-term change in teacher learning and practice.

To fully understand the impact of PD on teacher learning, it is important to look at both short-term changes in teacher' knowledge and practice immediately after the PD and long-term generative change that occurs several years out. For high-quality PD to have a lasting impact on mathematics instruction, gains from PD need to be sustained after the support ends. Furthermore, more research is needed to better understand the aspects of PD that have the potential to impact students and their learning. For instance, does providing resources and materials play a role? Does the degree to which one learns the content have a lasting effect? Or is a pedagogical strategy an impetus for long term change? Studying the impact of PD on teaching is a complex endeavor, intermingling the constructs of what is the nature of the impact (if there is one) and why there may or may not be an impact. More needs to be known about the ways in which teachers sustain their learning and how the learning unfolds several years after the PD. This study sought to examine what aspects of ambitious mathematics teaching were related to PD and sustained over time and why. After a cursory data analysis, we hypothesized that the use of visual representations (VRs) in PD may play a role and sought to better examine that. While there is evidence that VRs can improve student learning (Boonen, et al., 2014), less is known about the role they play in teacher learning and PD. This study explored the following research questions: In what ways do VRs play a role in teachers' learning and instructional practice?

In what ways do VRs learned in PD support the implementation of ambitious mathematics practices?

LITERATURE REVIEW

Ambitious Mathematics Teaching

Mathematics instruction that aims to develop all students' conceptual understanding, procedural fluency, reasoning, and problem solving is often referred to as ambitious mathematics teaching (Lampert & Graziani, 2009; Lampert et al., 2010). Ambitious teaching requires viewing students as sense-makers; eliciting and responding to students' thinking; and providing equitable access to learning mathematics. Ambitious teaching is complex and demanding as it requires continual learning about many things– the mathematics content, how to facilitate student understanding, how to foster engagement, and how to make learning meaningful for students.

The National Council of Teachers of Mathematics' (NCTM) *Principles to actions; Ensuring mathematical success for all* (2014) provides additional insight into ambitious teaching. *Principles to actions* (PtA) identified eight Mathematics Teaching Practices that "represent a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics" (NCTM, 2014, p. 9). These practices include (1) establish mathematics goals to focus learning, (2) implement tasks that promote reasoning and problem solving, (3) use and connect mathematical representations, (4) facilitate meaningful mathematical discourse, (5) pose purposeful questions, (6) build procedural fluency from conceptual understanding, (7) support productive struggle in learning mathematics, and (8) elicit and use evidence of student thinking. This framework offers a lens to examine instruction that supports successful mathematics learning.

Understanding how to support teachers' development of these ambitious teaching practices may be of interest to mathematics coaches, PD providers, and teacher educators. A growing body of research has examined how PD can assist teachers in developing these ambitious teaching practices, such as noticing and analyzing students' mathematical thinking and understanding (Fauskanger & Bjuland, 2019; Kazemi et al., 2009; van Es & Sherin, 2008; Wæge & Fauskanger, 2021). While these studies have shown that PD that connects the abilities of teachers and the actual work of teaching is important in developing ambitious teaching practices, what is less known is how these ambitious teaching practices change and are sustained years after the support ends.

Teacher Learning

Overarching goals for mathematics teachers' learning include improving knowledge of the content they teach, better understanding of student thinking and learning, and improving instructional practices to meet the needs of diverse learners (NCTM, 2014). Ball and colleagues define mathematical knowledge for teaching (MKT) as the complex set of knowledge needed to effectively teach mathematics to learners (Ball & Bass, 2000; Ball et al., 2005; Ball, et al., 2008; Hill & Ball, 2004). MKT is multi-faceted and includes both content and pedagogical knowledge and provides the field with a framework to focus on in PD (Jacob et al., 2017).

While substantial research has been conducted to examine the effectiveness of mathematics PD programs on developing teachers' MKT (Copur-Gencturk et al., 2019; Hill & Ball, 2004; Polly et al., 2014), less is known about whether this effect persists or continues to grow after the completion of the programs. Some studies that have examined long term uptake have found that after the completion of the PD, teacher knowledge and practice is sustained or continues to improve. For example, one study reported that mathematics and science teachers' use of inquiry- based teaching practices were sustained during the three years following their PD experience (Supovitz et al., 2000) and another showed that teachers' use of students' mathematical thinking in classroom observations was maintained or continued to grow four years after PD ended (Franke et al., 2001). However, others found that not all teachers sustain what they learned from PD and that initial changes in practice fade over time (Boston & Smith, 2011). Some helpful insights come from Copur-Gencturk and Papakonstantinou's (2016) longitudinal study of a Math and Science Partnership Program for high school mathematics teachers, an effort intentionally designed to incorporate key features of high-quality PD. The researchers followed participants for four years and documented linear instructional growth in several of the targeted areas. While teachers made statistically significant changes in some areas of their instruction, such as mathematical discourse, instructional clarity, and the development of students' mathematical habit of mind, over time, teachers were less likely to incorporate multiple representations in their classrooms despite the PD's focus on this. Given these mixed results in what is taken up and used over time from PD, more needs to be known about why this variation exists.

Visual Representations (VRs)
VRs are graphic creations such as diagrams or drawings that illustrate quantities, quantitative relationships, or geometric relationships (DePiper & Driscoll, 2018). Using models or representations is an important component of doing mathematics as they support students to make sense of problems by identifying quantities and the relationships between quantities and justify mathematical solutions (Ng & Lee, 2009). When students learn to represent mathematical ideas and make connections between them, they demonstrate deeper conceptual understanding and problem-solving capabilities (Fuson et al. 2005; Lesh et al., 1987). The use of VRs in the classroom also helps students reason mathematically and engage in mathematical discourse (Arcavi, 2003; Fuson & Murata, 2007; Stylainou & Silver, 2004). The use of VRs by mathematics teachers is complex and requires challenging skills including having a strong grasp of the content, anticipating students' thinking, and selecting the most appropriate VRs to use with students (DePiper & Driscoll, 2018). Teachers need to recognize what is involved in using particular representations and when they are appropriate to use (Ball et al., 2008). PtA outlines specific teacher actions that can support students in using and connecting representations such as: (1) introducing forms of representations that can be useful to students, (2) asking students to make math drawings or use other visual supports to explain and justify their reasoning, and (3) designing ways to elicit and assess students' abilities to use representations meaningfully to solve problems.

Research on teacher's knowledge and use of VRs in the classroom has been limited (Dreher & Kuntze, 2015; Stylianou, 2010). Researchers have found that teachers struggle with their own use of VRs to solve problems as well as their ability to use and interpret them in the classroom (Dreher & Kuntze, 2015; Orrill et al., 2008). Many teachers are also unaware of key instructional issues when using representations (Bossé et al, 2011; Dreher & Kuntze, 2015).

As will be described further below, both PD programs in this study included a focus on VRs. Participants learned about mathematical content using VRs and were exposed to different pedagogical strategies that involved the use of these representations. We were curious how this focus on VRs was related to teachers' uptake of not only their use of VRs, but also other ambitious teaching practices four to five years after the PD ended.

Overview of the Taking A Deep Dive (TaDD) project
This paper highlights a project that is part of a larger three-year impact study, Taking a Deep Dive (TaDD)[1], that collected qualitative data from three large U.S. National Science Foundation (NSF) PD projects to understand what teachers take up and use as well as the factors that influence uptake four to five years after the PD experience. This paper focuses on two of the PD projects, Learning and Teaching Geometry (LTG)[2] and Visual Access to Mathematics (VAM)[3], that aligned with recommended effective practice and were designed to support teachers in ambitious mathematics teaching. Table 1 provides a summary of the two projects, which are then described in further detail.

Table 1 *Overview of PD Projects*

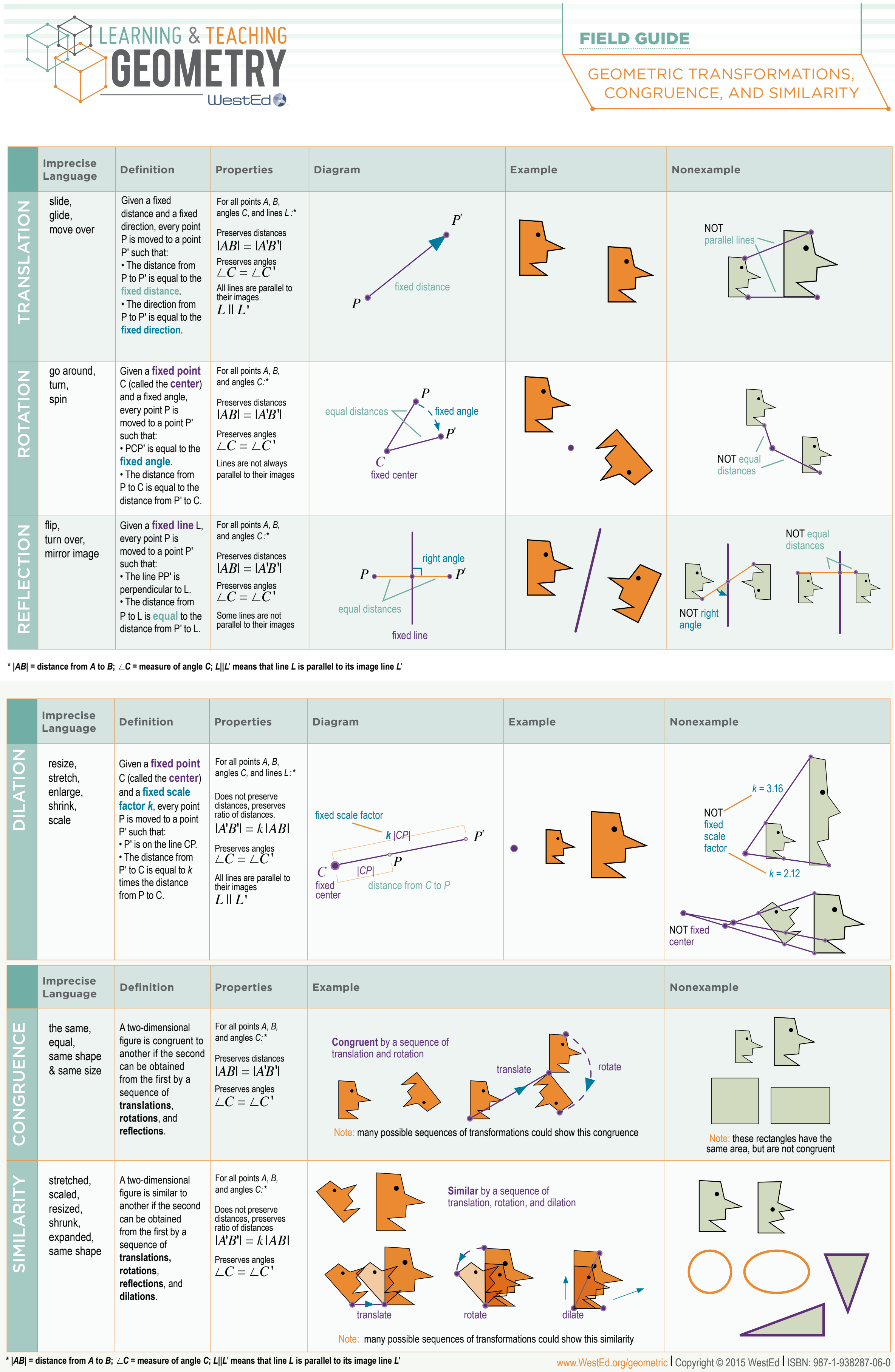
PD Components	LTG	VAM
Sample size	90 participants	120 participants
# of Hours	54 hours of in person PD	30 hours of in per-son PD & 32 hours of online PD
Content Focus	Similarity Transforma-tions-based geometry	Rational numbers in the middle grades
Pedagogical Focus	Visual Representations Classroom Discourse	Visual Representations Support for multilingual learners

Learning and Teaching Geometry (LTG)
The first NSF project, LTG, an efficacy study of the Learning and Teaching Geometry Professional Development Materials: Examining Impact and Context-based Adaptations, sought to study the impact of the PD on teacher's knowledge and instructional practices. The goal of the LTG project was not only to improve teachers' conceptual content knowledge and increase their ability to engage students in mathematical practices but to also increase students' conceptual unde standing of transformations-based geometry. LTG consisted of 54 hours of video-based PD that was grounded in modules focused on dynamic transformations-based geometry which is aligned with the Common Core State Standards for mathematics (National Governors Assocation Center for Best Practices & Council of Chief State School Officers, 2010). Along with examining classroom videos, teachers worked together to solve problems and further their knowledge in mathematics teaching in the domain of geometry. The PD allowed teachers to better support students in their attempt to gain a deeper understanding of transformations-based geometry through activities like rate of change on a graph, scaling activities, and similarity tools. The material strongly connects to other critical domains including similarity, proportional reasoning, slope, and linear functions.

The LTG PD highlighted various representations of geometric transformations, congruence, and similarity. All participating teachers received an illustrated glossary called the field guide at the beginning of the PD, which provided definitions, properties, corresponding diagrams (with examples and nonexamples), and imprecise language examples for terms *translation*, *rotation*, *reflection*, *dilation*, *congruence*, and *similarity* (see Figure 1). This resource

1. This work was supported by the NSF under Grant No. 1812439 2. This work was supported by the NSF under Grant No. 1503399 3. This work was supported by the NSF under Grant No. 1503057

Figure 1 *LTG Field Guide*



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FIELD GUIDE

GEOMETRIC TRANSFORMATIONS, CONGRUENCE, AND SIMILARITY

	Imprecise Language	Definition	Properties	Diagram	Example	Nonexample
TRANSLATION	slide, glide, move over	Given a fixed distance and a fixed direction, every point P is moved to a point P' such that: • The distance from P to P' is equal to the **fixed distance**. • The direction from P to P' is equal to the **fixed direction**.	For all points *A*, *B*, angles *C*, and lines *L*:* Preserves distances $\lvert AB\rvert = \lvert A'B'\rvert$ Preserves angles $\angle C = \angle C'$ All lines are parallel to their images $L \parallel L'$	P, P', fixed distance		NOT parallel lines
ROTATION	go around, turn, spin	Given a **fixed point** C (called the **center**) and a fixed angle, every point P is moved to a point P' such that: • PCP' is equal to the **fixed angle**. • The distance from P to C is equal to the distance from P' to C.	For all points *A*, *B*, and angles *C*:* Preserves distances $\lvert AB\rvert = \lvert A'B'\rvert$ Preserves angles $\angle C = \angle C'$ Lines are not always parallel to their images	equal distances, P, fixed angle, P', C, fixed center		NOT equal distances
REFLECTION	flip, turn over, mirror image	Given a **fixed line** L, every point P is moved to a point P' such that: • The line PP' is perpendicular to L. • The distance from P to L is **equal** to the distance from P' to L.	For all points *A*, *B*, and angles *C*:* Preserves distances $\lvert AB\rvert = \lvert A'B'\rvert$ Preserves angles $\angle C = \angle C'$ Some lines are not parallel to their images	right angle, P, P', equal distances, fixed line		NOT right angle; NOT equal distances

* |*AB*| = distance from *A* to *B*; ∠*C* = measure of angle *C*; *L*||*L*' means that line *L* is parallel to its image line *L*'

	Imprecise Language	Definition	Properties	Diagram	Example	Nonexample
DILATION	resize, stretch, enlarge, shrink, scale	Given a **fixed point** C (called the **center**) and a **fixed scale factor *k***, every point P is moved to a point P' such that: • P' is on the line CP. • The distance from P' to C is equal to *k* times the distance from P to C.	For all points *A*, *B*, angles *C*, and lines *L*:* Does not preserve distances, preserves ratio of distances. $\lvert A'B'\rvert = k\lvert AB\rvert$ Preserves angles $\angle C = \angle C'$ All lines are parallel to their images $L \parallel L'$	fixed scale factor, $k\lvert CP\rvert$, P', $\lvert CP\rvert$, P, C, fixed center, distance from C to P		NOT fixed scale factor, *k* = 3.16, *k* = 2.12; NOT fixed center

	Imprecise Language	Definition	Properties	Example	Nonexample
CONGRUENCE	the same, equal, same shape & same size	A two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of **translations**, **rotations**, and **reflections**.	For all points *A*, *B*, and angles *C*:* Preserves distances $\lvert AB\rvert = \lvert A'B'\rvert$ Preserves angles $\angle C = \angle C'$	**Congruent** by a sequence of translation and rotation; translate; rotate. Note: many possible sequences of transformations could show this congruence	Note: these rectangles have the same area, but are not congruent
SIMILARITY	stretched, scaled, resized, shrunk, expanded, same shape	A two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of **translations**, **rotations**, **reflections**, and **dilations**.	For all points *A*, *B*, and angles *C*:* Does not preserve distances, preserves ratio of distances $\lvert A'B'\rvert = k\lvert AB\rvert$ Preserves angles $\angle C = \angle C'$	**Similar** by a sequence of translation, rotation, and dilation; translate; rotate; dilate. Note: many possible sequences of transformations could show this similarity	

* |*AB*| = distance from *A* to *B*; ∠*C* = measure of angle *C*; *L*||*L*' means that line *L* is parallel to its image line *L*'

was used during their work on the mathematical tasks and warm ups. In addition, applets were used in the PD to display important mathematical content within an interactive, dynamic representation. In addition to a focus on representations, the PD also addressed additional ambitious mathematical teaching practices as teachers discussed facilitation of the tasks and watched classroom videos. Teachers reflected and discussed the mathematical content, representaions, student thinking and discource in the classroom videos.

Visual Access to Mathematics (VAM) PD

The second NSF project, VAM, was a "60-hour blended, face to face and online course to build teachers' knowledge of and self-efficacy about linguistically responsive teaching (LRT) strategies to strengthen multilingual learners' problem solving and discourse in middle grades" (Neumayer De Piper et al., 2021 p. 491). The goals and intentions of the VAM project were to cultivate in teachers the fluent use of representations, anticipation of students' strategies, the ability to interpret and construct various mathematical solutions, and to reason within and across representations. Teachers learned how to strategically select and align VRs with their instructional goals, anticipate student thinking and misconceptions, and then implement lessons using these strategies in their classrooms. Once implemented they would share experiences and student work, and collaboratively and independently reflect on the teaching cycle in the PD's online workshops.

VAM focused on two VRs, the double number line (DNL) and tape diagrams (See Figures 2 and 3). Both VRs are effective tools that have the potential to foster students' understanding of proportional reasoning and reinforce students' conceptual understanding of rational numbers (DePiper & Driscoll, 2018). The DNL is a representation that uses a pair of parallel lines to represent equivalent ratios. Tape diagrams, also referred to as bar diagrams, are rectangular representations that illustrate number relationships. Both diagrams represent quantities and the relationships between quantities. These diagrams allow students to "see" multiplicative relationships and examine the relationships between quantities with the representation. The VAM PD focused on problem solving with rational number tasks that were easily represented on a DNL or tape diagram. Subsequently, these VRs were used as a communication tool to show and explain students' mathematical thinking in a very concrete and conceptual manner.

The VAM PD also focused on additional ambitious mathematical teaching practices as participants engaged in tasks, planned lessons, and reviewed student work. In particular, the PD focused on LRT strategies to facilitate multilingual learners' mathematical problem solving and discourse. Participants learned about and experienced these different strategies and their implementation, planned the use of the strategies, and then analyzed and reflected on their implementation using student work.

Figure 2 *Double Number Line*

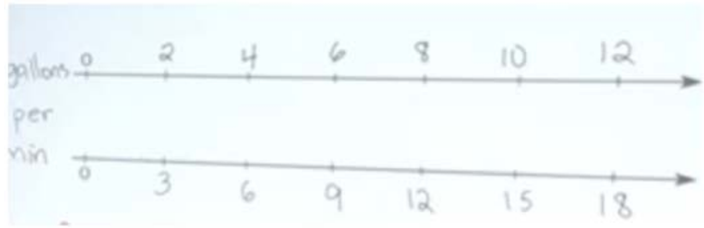
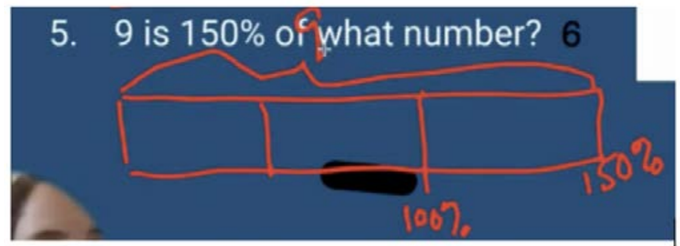


Figure 3 *Tape Diagram*



METHODS

Recruitment

As part of our larger study, project investigators from the VAM and LTG projects reached out to all participants from their respective projects four years after their participation in the PDs, to support our recruitment efforts to survey and interview participants from their projects. There was no PD support provided after the two projects ended and the only contact the PD providers had with the teachers was to support our recruitment.

Case Selection

Subsequently, the TaDD team used a survey to better understand participants' experiences (Koellner et al., 2022) and to select case study teachers. This survey included both closed and open-ended questions that asked participants to reflect on their PD experience and characterize their past and/or current use of the PD content, pedagogy, and materials as well as the support they received to implement new content and instructional practices. The survey included seven Likert scale questions, where participants responded to statements on a scale of 1-10, as well as 18 follow up questions that allowed the participants to explain and provide more details about their numeric response.
Using the responses to the Likert scale questions, we created an ordered list of participants from low to high uptake scores and divided the total in thirds, thereby creating three intervals. We then calculated the percentage of participants in each interval by project. We used the proportion of participants in each interval to select the number of participants from each category in order to fully capture the uptake by group as well as by interval. When selecting participants, we incorporated diversity of contexts with the averaged scores. Some participants elected not to be a case study teacher so we then selected the participant with the next closest score. We used this process until all participants were selected.

Ultimately eighteen case study teachers were selected based on self-reported levels of uptake (high, medium, low) from the survey. We asked them to videotape their classroom approximately once a month and identify clips in which they believed reflected content, pedagogy, and/or resources from the PD they participated in. The TaDD research team also conducted think aloud interviews along with the videos which will be described in greater detail in the following section.

In our review of the case study teachers' data for the TaDD project as a whole, we first investigated teachers with high levels of self-reported uptake on the survey, subsequently we moved to teachers with low levels of uptake and finally medium levels. We read the interview transcripts and watched video clips and took notes about the uptake of content, pedagogy, and resources from the PDs in order to familiarize ourselves with the data.

At least two of the research team members took detailed notes on the interview transcripts and classroom video data several times to create a profile for each teacher. Profiles included information about the case study teachers' contexts and backgrounds, relevant excerpts from the interview related to uptake of content, pedagogy, and resources and what they attributed their uptake and implementation to, as well as detailed notes about the video clips.We recognize that these teachers continue to hone their practice, attend different PD workshops, and attend to different goals of the school or district. We tried to account for these ongoing learning experiences to understand whether they have supported similar learning goals and objectives of the original PD as well as how the totality of experiences has supported productive teacher learning. We did this by asking specific questions about the other PDs they have attended and then asked them to pinpoint the origin of the content, pedagogy, or resource in the interviews. However, whether or not the PD was the sole contributor to a change in practice is not the goal here, rather we see the PD experience as one niche in the larger ecosystem regardless if it was the impetus for a new practice or supporting a burgeoning practice.

We conducted this first level of analysis using the profiles of our high uptake case study teachers. Specifically we coded the profiles of each teacher for the use and discussion of content, pedagogy, and resources that was aligned with the respective PDs that attended. There were two teachers from two different PDs who appeared to use representations more often in their teaching and also discussed the impact of VRs on their learning. We were curious why these two teachers that attended different PDs discussed the importance of representations, provided evidence of the frequent use of representations in their teaching, and attributed the use of VRs to their own learning from their respected PD experiences. This was basis for this study, to further investigate the role of representations in teacher learning and teaching as well as understand PD uptake by studying these two case study teachers— one from the LTG PD and one from the VAM PD.

The two teachers were selected as illustrative cases (Stake, 1995) of teachers that provided evidence of high levels of uptake from a PD four to five years post participation but moreover, these are two cases of teachers that illustrate the importance of VRs in teacher learning and how the different PD experiences supported and enhanced their instructional practices. The research questions that were the focus of this study are: In what ways do VRs play a role in teachers' learning and instructional practice? In what ways do VRs learned in PD support the implementation of ambitious mathematics practices?

Data Collection

For the purposes of this study, we used the qualitative data collected (interviews and classroom videos). Each participant videotaped six lessons between January 2021- December 2021. The teachers time-stamped clips in their video where they felt they were using content, pedagogy, or resources they learned from the PD. The videos were watched by the research team before the interview and notes were taken to inform our interviews. The teachers explained the video clips to the interviewers and how they attributed learning from the time-stamped clip to the PD. We also watched the classroom videos after the interviews and took notes again to better understand the teachers' reflections on them. The teachers were not observed live in their classrooms.

The TaDD research team conducted four semi-structured interviews with each case study participant (two in Spring 2021 and two in Fall 2021). Each interview took approximately one hour. The first part of these interviews asked teachers to reflect on their experiences with the PD, what they remembered related to the goals and intentions of the PD and what strategies, content, and resources they used from the PD in the past and continue to use currently in their classrooms. The second part of these interviews followed a think aloud protocol (Charters, 2003), where teachers and researchers watched video clips that the teachers selected. After playing the selected clips, the video was paused and teachers were asked explicitly to describe how the clip demonstrated uptake and implementation of content, pedagogy, and resources from the PD. The use of video allowed teachers to reflect on their practice and describe how they perceived their uptake in specific contexts and how they attributed specific learning to the PD. Moreover, they were asked in the interview to explain whether the learning started with the PD or whether the PD supported something they learned in another PD or something they were currently working towards. We recorded the interviews on Zoom and had them transcribed.

Data Analysis

We used a multiple-case study design (Merriam, 2002) to analyze the ways in which VRs and subsequently ambitious mathematical teaching practices were taken up and used in each of the teacher's individual contexts and how the teachers attributed this use to the PDs they attended. After initial cursory analyses the research team recognized that the VRs learned in their respective PDs were playing an important role in the case study teachers' classrooms. Thus, in year two, we intentionally asked probing questions to all participants related to their uptake and use of representations that were originally learned in their respective PDs to get a deeper understanding of the relationship between representations and their long-term learning. For example, we asked them, "What representations do you remember learning in the VAM/LTG PD? How did you learn about them?" and "What representations do you continue to use in your classroom and why?"

To answer our research questions, we used the profiles and then analyzed segments from video or interview data that related to how participants took up and used VRs and the ways in which they attributed this use to the PD they attended. Specifically from the interview data in the profiles, we took notes on how the teachers described their use of the VRs in the clips and the ways they attributed their use of the VRs to their PD. We also reviewed our notes on the classroom videos to ensure the way the teachers were describing their use aligned with what we saw in the videos. We also recorded any additional examples of representations in the videos that the teachers did not mention in their videos. Our initial codes related to VRs included: allocating time for students to use VRs, students discussing and making connections among VRs, introducing VRs that can be useful to students, asking students to use VRSs to explain and justify their reasoning.

We noticed that representations were playing an important role in the implementation of *other* ambitious mathematical practices, so we went back and recursively analyzed and coded the segments for other practices, such as implementing tasks that promote reasoning and problem-solving. At least two researchers independently coded these segments for evidence of the eight PtA Mathematics Teaching Practices. For example, a researcher might have coded a DNL task from a classroom video as promoting reasoning and problem solving or a researcher might have coded an example where the teacher facilitated meaningful discourse or encouraged productive struggle in a lesson using a ratio table (see Appendix A for a table of codes and exemplars). We used the triangulation of multiple data sources across time and multiple researchers to address issues of validity and credibility (Creswell & Miller, 2000). After examining these examples, the members of the research team used a thematic analysis (Braun & Clarke, 2013) to identify themes and patterns related to our research questions. The themes that emerged were: (1) the teachers' attributed their use of VRs to their own active learning of VRs in PD, (2) VRs appears to be a key factor that supported the teachers' use of other ambitious teaching practices in their classroom, and (3) the two teachers remembered and continued to use ambitious practices and VRs in their classrooms in ways that not only aligned to the goals and intention of the PD, but also adapted and extended representations to different mathematical domains and settings.

Next, the research team wrote narratives for each teacher related to these themes. Findings are reported as a case for each teacher. Each illustrative case includes specific examples related to how VRs impacted their own learning as well as evidence to demonstrate how each teacher used VRs connected to PtA Mathematics Teaching Practices which we believe are examples of ambitious mathematics teaching. Additionally, we highlight how these examples are related to their experiences in PD and how they continued to modify and adapt the use of these representations over time.

TEACHER CASE STUDIES

TEACHER #1: BRIANNA

Brianna took part in the LTG PD. She had taught for 11 years at the start of our study and had taught grades 3-8 mathematics. Throughout this research project, Brianna taught 6th and 8th grade mathematics in a middle school in the Western United States. She attended the PD because she was teaching geometry, and since she had recently moved to teaching middle school mathematics she was interested in learning and teaching transformations-based geometry. Brianna was chosen as an illustrative case study as she demonstrated high levels of uptake compared to other teachers who attended the LTG PD in her survey data, In addition, relative to other case study teachers, her classroom video data contained many examples of the use of VRs and she discussed her learning about and uptake of VRs more often in her interviews. Brianna's experience in the PD is described below and four examples are provided that illustrate her sustained use of VRs in her classroom that she attributed to participation in the PD four to five years after participating in the LTG PD. The examples demonstrate how her use of VRs sometimes cut across different mathematical domains and tasks compared to those used within the PD. Each example also illustrates how the use of the representations is connected to other ambitious teaching as defined by the effective teaching practices outlined in PtA.

Experience in PD

Brianna discussed how participating in the PD allowed her to deepen her conceptual understanding of the mathematics content that the PD focused on (geometry). She also felt that it was helpful to work through the math problems as students would during the PD. She mentioned in one of her interviews that took place four years after attending the PD:

> *It was really helpful for me to gain a better insight of the math that I was teaching…to be a student, to learn how to better understand all of these ideas in a way that's more conceptual than what I learned as a student. So that's how I felt like it was most helpful to me.*

She also mentioned that, in addition to learning new mathematics content this way, it also exposed her to new pedagogical skills to teach the content, such as providing patty paper to foster an understanding of transformations. She felt an important component of the PD was the opportunity to view a lesson from a different perspective and put herself "in the shoes of my students."

Example #1: Number Lines

In her first interview, Brianna was teaching remotely due to the COVID-19 pandemic. One of the video clips she chose to discuss with us was a synchronous lesson about negative and positive integers. In the lesson, students were shown vertical number lines and encouraged to use them to think about changes in temperature. She then used an online platform (PearDeck) to have students solve various problems with the vertical number line. The online platform allowed her to choose different student strategies all using the vertical number line and then facilitate a discussion with the class. Her interview and classroom video showed many of the effective teaching practices from PtA but especially highlighted the use of tasks that promote reasoning, use mathematical representations, facilitate discourse, and elicit student thinking.

After she shared a video clip of the vertical number line lesson with us in the interview, she discussed how this lesson connected to her learning from the PD:

> *One of the things that was really important in the PD was the use of models. And while a lot of those models were on coordinate grids and graphing and shapes and scaling and whatnot, these that I used in this lesson were number lines that we were using to be able to go from negative to positive numbers. And the problems and the number lines also required students to be really precise with how they were measuring and how they were representing temperature change in their number lines.*

Her reflection on the clip indicates that the use of models or representations was something she felt was an important component of the PD. She also explained to us how she utilized VRs in her classroom practice regularly which was confirmed by each of her classroom videos. She felt strongly that representations supported access and student learning. In her video about the vertical number line, we saw her engage in several mathematical practices related to the use of VRs. She introduced a representation that can be useful to students (number lines), and she asked students to use their number lines to explain and justify their reasoning. We also saw how the use of VRs helped her facilitate meaningful discourse by having students examine and discuss each other's representations. In addition, although the PD focused on representations in transformation-based geometry, she generalized this use of VRs and the need for precision in a representation to a topic that was not discussed in the PD: operations with rational numbers.

Briana also noted how the representations allowed her to have students view one another's work and learn from one another, which was a teaching practice that was important to her:

> *What I did was, I could select a couple and then I could show some exemplars. So kids could see at the beginning what some exemplar work looked like. So rather than me showing them how to do it again they could see from their peers and then the rest of the problems that they did more independently - they had a good starting point for how to do those.*

Her decision to use a vertical number line representation related to several ambitious teaching practices she felt were important for instruction and related to the PD: selecting tasks that promote reasoning, using mathematical representations, facilitating discourse by engaging students in purposeful sharing of varied representation and eliciting and using evidence of student thinking.

Example #2: Dot Images

In another example, Briana shared a clip of how she used representations to begin a unit on writing equations. Students were given the following task: *I'm going to show you an image made up of dots, but only for three seconds. You need to find out how many dots there are and be prepared to explain how you saw them*. After they were shown the image, she told them, "Before we talk as a whole group, I'm going to have you meet with your two-by-two partner. I want you to say how many dots you saw, and then explain how you saw them." Students then discussed what they saw in the image with their partner, and then Briana selected students to share what their partner said with the whole group.

Briana explained to us how she felt this clip demonstrated her implementation of ideas she learned in the PD. She connected this task to her learning from a video used in the PD that included VRs of transformations, where students shared what they noticed about the different images. That video from the PD resonated with her years later and, although she was teaching different content, she utilized a representation and similar teaching practices to facilitate discussion and allow students to share their thinking. Below is her description of how this example connected to the PD:

> *I chose this one because I remember when I took the geometry class (PD), I feel like they were on an overhead projector, but the teacher's teaching in, I think it's Hawaii.*
>
> *They're looking at something and all of these kids just keep sharing their answers, and it's this very open classroom environment where kids just get to talk and share and they're "oohing" and "aahing" about what they're noticing. And so, this was the opening to a lesson and we're talking about factors and multiples, and so they're subitizing. They're looking at different dot structures to figure out how much they are or how many are there.*

We see in her comments and in the videos that she viewed the dot image as a vehicle for her to facilitate meaningful discourse and allow students to present and explain their ideas.

She also discussed another important idea from the PD about connecting mathematical concepts and using a warm-up task to help create an entry point for learners:

> *This goes back to when I was talking about being thoughtful about the type of warmup that I'm doing, that it's this fun task that relates to what we're going to do later. And it goes from just, "How many dots do you see?" to being able to write an equation. So there's this very easy (task), all the way to something that's more difficult. But if I just started off by saying, "Hey, write an equation for this," they'd be like…*

> *"How do I do that? What does that even mean?" So getting them to that point of something that's much higher-level thinking through, something that's kind of fun.*

In this example, Briana highlights how her use of a visual representation is connected to the PtA effective teaching practice of implementing tasks that promote reasoning and problem solving. Her choice of the dot image task provided students with multiple entry points to the mathematical concepts, allowed students to build on and extend their current mathematical understanding and encouraged students to use a variety of approaches and solutions. Finally, we see that while this task was not one used in the geometry PD, she generalized the ideas she saw in the PD and applied them to the content she was currently teaching.

Example 3: Ratio Table

Briana also chose to share several clips of her teaching with ratio tables with us. Again, we were curious how she felt this connected to the geometry PD. She explained to us how this was related to the scale factor content that had been addressed in the PD:

> *This school year we did so much work with ratio tables. And while it's different from the scaling up and down work that we did in the PD, it really helped me to say, okay, these are the foundations that I want to set for my kids to be able to do that work later on in two years. So that was another piece that was really helpful as well.*

Again, we see how Briana took ideas from the PD and transfers them to other content areas. She also described how she felt that the introduction of these ratio tables in sixth grade would prepare students for the content that had been discussed in the PD when they entered eighth grade. This reflects her understanding of mathematical learning progressions and points to her ability to establish mathematical goals to focus learning (PtA Mathematics Teaching Practice #1) by identifying how the goals of a lesson fit within a mathematical learning progression (Confrey, 2012). She viewed the ratio table representation as a tool for her to build important mathematical understanding for both the current content and the content students would be exposed to later.

In her classroom video clips she shared multiple lessons where students used ratio tables to compare rates and ratios. In her think aloud protocol during the interview, she talked about how she viewed this representation, ratio tables, as a tool that allowed students an entry point to solve challenging problems. She viewed the ratio tables a tool to help students develop conceptual understanding of ratios and indicated that she saw the ratio table as having multiple uses across different types of problems.

> *The sixth-grade teacher and I do many things with ratio tables. We're very much like, we want them to have that tool to be able to use, so we do so much work with our ratio tables. By sixth grade, our goal is not really for them to have an algorithm, but to be able to just be super-efficient with their ratio tables. And we talk to kids a lot about how the whole goal is that all of your work is on your ratio table. This is the tool that you're using. So you're not doing all this work off to the side, but this is the tool that you use. And then kids get really good at it, and it also helps their mental math, so that then they'll start to solve other problems like, "Oh, I can just think about that in my head this way."*

In this example, Briana highlighted how she used a visual representation as a tool to implement tasks that promote reasoning and problem solving (PtA Mathematics Teaching Practice #2). The use of the ratio table provided students with multiple entry points to the mathematical concepts and allowed students to build on and extend their current mathematical understanding.

Example 4: Polygon Sort

In this final example, Briana shared a video of a sorting task she used in which students explored examples and non-examples of various polygons. Students sorted shapes into categories with a partner. They discussed the characteristics of the different figures and Briana supported them with using precise vocabulary to describe the shapes and creating definitions. When asked how this lesson connected to her learning from the PD, she explained how she modified transformation tasks from the PD to this content:

> *[In the PD]...they would say things like, here's your original and here's your image. And we would talk about which one of these would work. So it would have examples of, "Well, this could be a translation." These could all be translations, but here's an example of something that is not a translation maybe because it's rotated too. And so to get kids to, not only am I going to notice one of these that's different, but I also have to support that with my thinking. So I have to go one step further to be able to explain what I observed and why I feel like it fits this set of rules or why I feel like it does not fit the set of rules.*

We see that Briana took the format of these transformation tasks and the visuals that they used and applied the ideas to a lesson on polygons. One idea from the task that she found salient was that it elicited students' thinking as they had to notice differences and that it encouraged them to justify their thinking to develop their understanding. This is consistent with PtA Mathematics Teaching Practice #8: Elicit and use evidence of student thinking. The sort with the VRs of polygons allowed her to elicit and gather evidence of student thinking at strategic points in the lesson. We also saw her engaging in PtA Mathematics Teaching Practice #2: Implement tasks that promote problem solving and reasoning. The sorting task allowed students to use representations to make sense of the mathematics and develop a conceptual understanding of what a polygon is. Finally, we see through these examples how she once again applied learning from the PD to a different content area with her sixth-grade students.

She also discussed a resource from the PD that she found very helpful and explained how she modified it for her students for polygons. As mentioned earlier, a visual one-pager called a "field guide," was given to teachers in the PD to support their understanding of transformations. She created a similar one for polygons and used this visual support for students in her classroom.

> *Yes. That field guide is, every teacher who teaches geometry should have that. And we've tried to emulate that with some of our other things that we teach. To have some sort of reference guide that kids can use. Because I kept this on the board for a few days and we continued to talk about polygons and you could see them, they would start to talk about polygons and then their eyes would look over. I know there's a reference sheet that is on board that I can use to help me, support my thinking.*

Briana took the idea of this field guide resource from the PD and had students co-create a graphic organizer for a different content area. She also highlighted the importance of visuals in both the guide from PD and in this resource the students create.

> *What we've made is more of a graphic organizer that has the things we want them to know, but then we fill it out together. So it's not just a blank sheet where they have to write down all the notes, but it has visual examples which were big in the field guide, but also opportunities we'll go through and we'll annotate different aspects of whatever we're thinking about…So the page it's about parallelograms has a picture of a parallelogram. It shows how a parallelogram can be rearranged into a rectangle. So there's a visual example at the top and then there's a couple practice problems for them to do. So I would say the top part of each page is a lot of the field guide.*

In these examples, we see how Brianna used the representations to implement tasks that promote reasoning, facilitate discourse, and elicit and use evidence of student thinking. The sorting task incorporated VRs, provided entry points for students, and helped develop their justification skills. The adaptation and creation of the graphic organizer provided visual support for students that they could use as they worked on tasks. She anticipated what students might struggle with and provided a tool that would support them. Briana provides a case of how a teacher who participated in a specific PD, LTG, took up and used VRs she learned about four to five years. She provides insight into the ways in which a teacher adapts and modifies what she learned in PD over time. In addition, by examining a teacher who had high levels of uptake compared to others, we can begin to understand the ways in which she attributed her use of VRs and mathematical practices to the LTG PD and how she continued to modify and expand her use over time.

TEACHER #2: RACHEL

Rachel took part in the VAM PD. She had taught for 11 years at the start of our study and has taught grades 5-8. Throughout this research project, which occurred four to five years after she participated in the original PD, Rachel taught 7th grade mathematics in the Northeast. She attended the PD because she wanted to learn new strategies and obtain new resources. Rachel was selected because, like Brianna, she demonstrated high levels of uptake in the survey and her classroom videos included multiple examples of VRs that she attributed to her learning from the PD in her interviews. However, she attended an entirely different PD than Brianna and we wanted to better understand how her PD experiences and continued use of VRs aligned with Brianna's. Rachel's experience in the PD is described below and three examples are provided that illustrate her extensive use of representations in the classroom years after she participated in the PD. Like Brianna, her use of VRs cut across different contexts and domains than those used within the PD. The examples presented also demonstrate how her use of these VRs was connected to other ambitious teaching practices, such as implementing tasks that promote reasoning and problem solving and facilitating meaningful mathematical discourse.

Experience in PD

Rachel described having a positive experience in the PD. When asked about what she remembered learning and continues to use, her answers focused on the importance of teaching with VRs. Prior to participating in the PD, she was unfamiliar with using the DNL and tape diagrams to solve ratio and proportion problems and had not used them in her classrooms. She mentioned in her interviews.

> *I didn't know a lot of the representations that they were teaching us [in the PD]. I had been teaching middle school math for 8 years and I had never used a DNL. I was solving these problems using proportions or equations and I never knew this thing existed.*

Prior to the PD, Rachel mentioned that she used equations to solve proportions, as she did not know about the VR options. This aligns with the research indicating that when teachers are unfamiliar with how to use these tools, they typically rely on algorithmic thinking to solve these types of ratio and proportional reasoning problems (Orrill & Brown, 2012). She also described how the PD helped her develop a conceptual understanding of topics, as well as helped her think about how these VRs could be used with her students to create access for different learners.

> *...it was really more just pushing me out of a procedural way of thinking about math. One example of that is one of the first days they asked us to make a visual model to represent some situation. I didn't know what that was so I solved it using algebra. Other people in the PD had experience using tape diagrams and double number lines and I didn't understand why they would do it that way. I thought the way I did it was so much easier. They told us to represent this using a tape diagram and a double number line which sort of forced me into thinking about how do I represent this same problem using this method and what are the benefits of presenting it this way and how can that help struggling learners or students that are not really able to access the curriculum because of language.*

After the PD, Rachel began to incorporate these representations in her teaching and continues to use them four to five years later. This was evident in both her interviews and classroom data provided. Each of the videos submitted was coded for the use of VRs and she mentioned them all of her interviews. She explained how one representation, the DNL, transformed her teaching.

> *I started to take the DNL and completely change the way I teach ratio and proportion and percent and I started to use the tape diagrams and the DNL for everything. I still am using the materials from VAM for those units.*

Rachel's exposure to and practice with different VRs in the PD seemed to have impacted her own understanding of how to solve ratio and proportion problems with representations, as well as provided her with new pedagogical strategies to use with her students. Three examples that demonstrate this uptake of these ideas are presented in the following sections.

Example #1: Unit Rate

Rachel shared a video with us related to unit rate and described how she used the DNL to provide access for students. The videos were collected during a synchronous zoom class. Students had previously worked on unit rate and in this lesson, she introduced them to the DNL as another way to think about the rate. They had worked with a warm-up that involved them figuring out the clicks per second of a robot. The students then created a DNL for this situation along with her and they discussed how to label it and use it to solve problems.

> *We had started unit rate. The day before this we did an activity in Desmos and it's robots clicking per second and that is how we got into the unit rate, that way they had an idea of what unit rate is. We had a way to jump into double number lines with something they were already kinda familiar with. That's kinda part of the PD because they talked about giving students experiences to relate to.*

In addition to discussing connecting the mathematics to students' lives, she highlighted the use of a particular representation (DNL) as a novel approach that she had not used prior to the PD.

> *But the biggest PD thing here is the double number line here because I would have never used it otherwise. Figuring out how to relate this double number with what we did for the warmup was kind of a VAM PD thing too, making a connection between what is going on in this situation and the visual representation of it. I can't remember what the specific warmup problem was but they were finding the unit rate of the clicks per second, the clickbot they called it. We were then showing the unit rate on the double number line of the clicks per second.*

She also discussed how the DNL representation allowed students to access the problem in different ways which is connected to implementing tasks that promote reasoning and problem solving (PtA Mathematics Teaching Practice #2). She described how the task and in particular the use of the DNL allowed for multiple entry points and helped students make sense of the problem posed. She explicitly connected this to a learning from the PD: "That is also part of the PD, being able to get every kid a way into a problem so that way everyone feels like they have a way to solve it."

The use of the DNL also allowed her to facilitate meaningful discourse with her students. PtA defines meaningful discourse as building a shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments. She shared a clip of the students in a breakout room discussing their solutions to a problem where they had used the DNLs to solve.

> *When I got into their breakout room, they were yelling at each other. Not yelling angry but they were both arguing what I thought was the same argument. The first kid that was talking couldn't put into words how many minutes he thought it was because he was getting tripped up by the fractions. The other kid knew it was a fraction or a decimal number but he called it a half because he didn't really know what else to call it either.*

In the video clip, we saw her go into the breakout room and ask the students to show their solutions on the DNL. She then asked them to describe their solution to one another using the DNL and then discuss what the correct answer was. She also explained how the DNL also helped them developed their understanding:

> *Using the double number for this helped them to be able to put a value on what that fractional value could be. This is one of those things that using the double number line makes it so much easier for the kids to kind of think about that instead of getting stuck. It means more when they are able to break it down and look between. It makes for a much richer problem solving experience rather than just going back and thinking about it as an equation or proportion to solve it. It is helping them*
>
> *break down the relationship between the two units, where the rates are.*

In this example, she discussed how the introduction of the DNL representation was a useful way to help students develop their conceptual understanding and problem-solving skills. She also focused on the role it played in allowing students to communicate their thinking. Rachel noted throughout her interviews that one of the ideas that resonated with her from the PD was the use of VRs as communication tools to engage students in purposeful sharing of their mathematical ideas (PtA Mathematics Teaching Practice #3). We saw several other classroom examples of her engaging in this ambitious teaching practice of facilitating mathematical discourse by asking the class to compare VRs and discuss what they noticed and what they would change about the representations (PtA Mathematics Teaching Practice #4). For example, in one video, she presented two tape diagrams that were created by two different students to solve a word problem. She asked students to compare and contrast the solutions and how the quantities in the scenario were represented in each diagram.

Example #2: Rational Numbers

Rachel shared with us an example of students using visuals for a lesson related to operations with rational numbers. Students had previously worked on a word problem about the temperature dropping a certain rate and many of them had gotten the problem incorrect. She began her lesson the next day by asking them to draw a picture of the following scenario: *The temperature is 24 degrees; it falls three degrees every hour for six hours. Draw me a picture of it.* Students then drew pictures and she strategically selected students to share their representations and discuss them. Rachel talked about how this was an example of how she continually encourages students to make representations as they work through problems:

> *Every day, I'm saying, "Can you draw something to represent this? Draw me a picture of what's happening? Can you show me on a number line? What is happening?" And every time I asked them to draw it out, all of a sudden, their question disappears. So, I wanted to make the point that if you draw it out you've answered the question before you even read it, and so I kept the question out of the warm up…so um the pictures all had the minus 18 and they all had the you know falling and they all knew where it was ending up, and so, then I was like, I didn't even ask you a question and you've answered like two different questions, right now, and this is why, when I when you ask me a question about a math problem or something I say draw a picture because chances are it's going to solve the problem before you even know what you're supposed to be answering so when you can model it out that helps you.*

Although this was not a specific representation discussed in the VAM PD, Rachel chose to share this clip as an example of how she took up and used something from the PD. We see that encouraging students to make representations of a situation is a way in which she helped students engage in reasoning and problem solving. It also connects to the teaching practice of supporting productive struggle (PtA Mathematics Teaching Practice #7) as she anticipated what students might struggle with and encouraged them to use representations as a scaffold for accessing the tasks.

Example #3: Expressions and Equations

In the final example, we see how Rachel took a task from a different curriculum and applied what she learned about representations and ambitious teaching practices from VAM to it. Students were working on an activity where they had cards and had to first sort and notice things about the algebraic expressions, equations and VRs and then match the equation or expression to the correct VRs. She talked about how she felt this video related to the PD:

> *We didn't get this from VAM, I believe it is from Open Middle Math curriculum…. Not from VAM but a VAM-y type problem where it is very open-ended at the beginning where they are sorting and noticing different things about the equation. Then they have to commit and say this matches this visual because of this and this. Drawing the connections between the equation notation and a visual model of what is actually happening.*

We see again how she selected and implemented a task that aligned with ambitious teaching practices she felt were important from the PD. In this case, she chose a task that promoted reasoning and problem solving (PtA Mathematics Teaching Practice #2) and had students first explore the VRs and what they noticed to provide an entry point for students. She then extended the task and provided them with the opportunity to connect algebraic expressions and equations to visual models, like tape diagrams. Although the task was not from the PD, she referred to it as a "VAM-y" type problem which demonstrates how she has generalized aspects of the tasks used in the PD and selected a task that aligns with these principles, such as providing access for students.

She also talked about how the visuals helped them develop a conceptual understanding of the equations:

> *For the most part they did pretty awesome, they got really tripped up with the 19/2=x+5. One of the equations was 2(x+5) and they got that really quick. The working backwards part they got all sorts of confused because 19 is not easily divisible by 2. Their little seventh grade brains were like "Can't do it! It's a decimal." We talked about how if it's 19/2 that means we are cutting nineteen in half. We don't need to think about it as 9.5. Think about it as taking half of the tape diagram. So, which of our choices is half the tape diagram equal to x+5. Once they saw it like that there were a lot of light bulb moments.*

In this case, she used the visual diagrams to help students make sense of the quantitative relationships in the equation. She also articulated other ways that the task allowed them to better understand the expressions and equations and stressed how this related to her goals of having them make sense of different ways to write equivalent expressions and equations. This example also demonstrates her ability to take up the learning about representations and ambitious teaching practices and apply them to a content area that had not been discussed in the PD: algebraic equations and expressions.

SUMMARY OF CASE STUDIES

The two illustrative case studies help us better understand the ways in which two teachers with high levels of uptake from PDs sustained that learning over time. Although the format and implementation of the two PD projects differed, similarities emerged between the uptake and residual learning related to ambitious practices and representations. Examining these two cases in depth allows us to better understand examples of high uptake across different PDs to begin to understand how and why some teachers sustain learning years after PD support ended. These similarities are detailed below.

The prevalence of the use of representations in the teachers' classrooms appeared to be strongly connected to teachers' own learning about VRs in PD, both in terms of content knowledge of how to use them to solve problems and pedagogical knowledge related to how to use them with students. Both teachers' use of representations and their explanations related to their choice to use representations is nontrivial. In other words, they did not simply use a representation in the exact way they had learned about it in their respective PDs. Rather, they were able to discuss the complex mathematical content they learned using a representation as well as make connections to other mathematical domains. They also discussed the ways students might use the representation and anticipated student strategies, both correct and incorrect using the representation. The representation seemed to be the catalyst in many or most instances. For example, Rachel talked about how her understanding of solving ratio problems changed because of seeing other participants in the PD use tape diagrams and DNLs to solve the problems and how this in turn caused her to think about how the use of VRs could provide more access for her students. This learning about new representations in the VAM PD changed the ways in which she taught her ratio unit as evidenced in her classroom videos.

We saw similar changes in Brianna as she mentioned how learning about geometry through representations allowed her to put herself in the shoes of students and then changed the ways in which she provided access through representations in her lessons. Thus, we hypothesize that a representation can be an important catalyst to teachers' mathematical learning, which also supports their pedagogical practice— typically using the representation in problem solving situations similar to their experience in PD whereby the representation becomes an important mediator for teachers' learning as well as their implementation of ambitious mathematical practices to support student learning.

Both teachers discussed how they used representations in their classroom and how they provided access for students, and they both highlighted how selecting different ways students used representations to solve the problem helped them to facilitate communication and discussions in their classrooms. Thus, this encouraged and supported their use of the mathematical practices or ambitious teaching practices. More specifically, some of the practices evidenced in video and discussed during interviews included using rich tasks that promote reasoning and problem solving, facilitate meaningful student discourse, promote productive struggle, and elicit student thinking. One example from Brianna's video was when she used a dot image task to provide an entry point into writing algebraic equations. Although this was not a task from the PD, she was inspired by the way the PD facilitator used VRs to provide entry points and facilitate discussion. This led to students sharing strategies, making connections, debating solutions, and using reasoning to justify their responses. It also allowed her to make connections to more complex content. Again, these results may suggest that learning about the VRs appears to be a key factor that supported teachers use of other ambitious teaching practices in their classroom.

The two teachers remembered and continued to use ambitious practices and VRs in their classrooms in ways that not only aligned to the goals and intention of the PD, but also adapted and extended representations to different mathematical domains and settings. They attributed their use of VRs to the PD itself and sometimes the PD helped to solidify their learning and support changes in their instructional practice to include more ambitious approaches. They highlighted and remembered specific teaching strategies and VRs from the PDs even though they had not received any intervening support from the PD providers in the four to five years since they participated in the PD. They also designed tasks for new content areas that incorporated VRs and other ambitious practices and adapted tasks and strategies from the PDs to their new online settings during the pandemic.

DISCUSSION

To understand the impact of PD on teacher learning, this study examined long-term learning, changes in pedagogy and potential generative change in participants four to five years post PD experience. The findings add new insights for teachers, teacher leaders, and administrators related to the importance of VRs for both teacher and student learning of mathematics, as well as the importance of studying teacher learning and the impact of PD over time. This study contributes to the literature by providing examples of how learning about a specific pedagogical tool, in this case VRs, along with specified content, can have an impact on teacher learning and pedagogy over time.

Our analysis indicated three main findings as described above: (1) the teachers' use of VRs appears to be strongly connected to teachers' own active learning of VRs and content in PD, (2) VRs appears to be a key factor that supported the teachers' use of other ambitious teaching practices in their classroom, and (3) the two teachers remembered and continued to use and hone ambitious practices and VRs in their classrooms in ways that not only aligned to the goals and intention of the PD, but also adapted and extended representations to different mathematical domains and settings.

The sustained use of VRs appeared to be strongly connected to teachers' own active learning in PD and their development of content and pedagogical knowledge which have been cited as components of effective PD (Desimone & Garet, 2015). Teachers, like students, used the VRs to make sense of new mathematical concepts, make connections among concepts and as a tool to communicate and share their thinking. This study provides evidence that there is some similarity between teacher learning using VRs and that of students' improved learning when using VRs (Boonen, et al., 2014). Furthermore, in both projects, the VRs used in the PD were intentionally selected to teach specified mathematical content. We hypothesize that PDs designed in this way, where teachers are learning new mathematical content with the use of a specified representation, have the potential to create robust learning which translates to practice and sustained use. Additionally, the use of VRs does not appear to be contained to the content area that was the focus of the PD as we have evidence that the case study teachers continued to use VRs and ambitious practices not only with the content that aligned with the PD they attended but also across contexts and domains.

This contrasts with PD programs where the use of multiple representations is introduced and encouraged but perhaps lacks the specificity of when and how VRs would best be used and with what aspects of the curricula. We wonder if this might be related to the fact that the case study teachers presented here learned both relatively new content to them with specific VRs to support their conceptual understanding. And in turn, we hypothesize that this also supports the sustained use of VRs to teach mathematics. This may also explain results from Copur-Gencturk and Papakonstantinou's (2016) study where the PD was a large endeavor focused on comprehensive content domains including geometry, linear algebra, and statistics and probability, and also featured a vast array of pedagogical techniques including a focus on discourse, formative assessment, habits of mind, and multiple representations. Thus, the goals and intentions of this large effort might actually be well intended and have success in more broad pedagogical areas such as discourse, but might not provide enough specificity for the selection and implementation of specific VRs to support both teacher and student mathematics learning. This might provide insight into why the researchers found statistically significant changes in mathematical discourse, instructional clarity, and the development of students' mathematical habit of mind over time, but not in the use of multiple representations.

This study also sought to examine if ambitious mathematics teaching practices that were related to the PDs were sustained over time and why. This study provides two illustrative case studies of teachers that continued to use ambitious mathematics practices. We reported on the importance of these practices to reach more learners, but what we noticed in particular was that much of the time these ambitious mathematical practices were used when they were tied to the use of VRs. VRs were the focus of many students' strategies and therefore the basis of important and rich mathematical discussions. Students used the mathematical representations to justify their thinking and make their point when explaining their solution to complex problems. The VR essentially became a mediator between the student and their mathematical thinking and the way in which they conveyed their understandings to the teacher and classmates.

This is somewhat similar to research on designed instructional activities (Lampert & Graziani, 2009; Lampert et al., 2010). Lampert and colleagues found that creating design activities supported and encouraged novice teachers to implement ambitious mathematics teaching. Similar to their thinking that the use of instructional routines may reduce the cognitive load of ambitious teaching, the use of VRs may also have the potential to serve a similar role. One difference might be the fact that teachers need to develop pedagogical content knowledge related to the content and the VR such as how and when to use VRs whereas routines are more prescribed. However, the case study teachers presented here seemed well equipped to implement VRs in different contexts possibly because of the relationship between their own learning of content with the support of the VR. Math education leaders may want to consider how VRs an be used as an anchor for developing both teachers' understanding of the mathematics and pedagogical strategies. Additionally, perhaps more research is needed to investigate routines or design activities that might support teachers to use specified VRs effectively or perhaps more research is needed to understand the differences between designed instructional activities and the use of dynamic representations and their alignment to ambitious practices.

This study also showed that studying teacher learning over time was important in our study as it allowed us to assess and learn that teachers continued to hone their pedagogical practices learned in PD over time. As Kennedy (2016) noted in a review of the literature, the impact of PD is not always visible immediately following participation. These cases demonstrate more evidence of ways teachers used learnings from the PD, what they attended to, and the different and generative ways they implemented mathematics instruction in other mathematical domains, with different problems, and with different grade levels.

Implications for PD providers, teacher educators, and school leaders suggest that a focus on VRs may be one tool to anchor learning in a PD to deepen teachers' abilities to engage their students in ambitious teaching practices. Some suggestions include:

- Exposing teachers to new VRs in professional learning. Teachers may benefit from learning about new VRs and using them to solve mathematical problems in PD to learn new content or to deepen their conceptual knowledge.
- The use of VRs in professional learning can be used as a tool to highlight mathematical practices and as a model to support their implementation of pedagogy to support ambitious teaching. PD facilitators can help teachers explore how representations can be used to facilitate meaningful discussion. For instance, as teachers grapple with how to select, sequence, and have students share different strategies in their class.
- The use of VRs in PD can also facilitate discussion with teachers about how the use of representations can provide multiple entry points to tasks that promote reasoning and problem solving as well as how representations can be used to support productive struggle.
- Even when content of PD is specific, including a focus on representations may allow participants to generalize to other topics and contexts. Both participants applied the use of representations to online contexts and to different mathematical domains that were not discussed during their time in the PD.
- This study focused on two teachers with high levels of uptake and provided some understanding into what these teachers took up and used. While we do not suggest that the interpretations found are generalizable for all teachers, these patterns begin to illuminate the

important utility and the role representations may play in the enactment of ambitious teaching practices. In addition, more needs to be understood about teachers that participate in PD and do not attain the same level of uptake. Future case study research is planned to begin to understand the differences in uptake among participants. ■

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REFERENCES

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52(3), 215-241. https://doi.org/10.1023/A:1024312321077

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Ablex.

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27-44). National Council of Teachers of Mathematics.

Ball, D. L., & Even, R. (2009). Strengthening practice in and research on the professional education and development of teachers of mathematics: Next steps. In R. Even, & D. L. Ball (Eds.), *The professional education and development of teachers of mathematics* (pp. 255-259). Springer.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*(1), 14-17, 20-22, 43-46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Boonen, A. J. H., van Wesel, F., Jolles, J., & van der Schoot, M. (2014). The role of visual representation type, spatial ability, and reading comprehension in word problem solving: An item-level analysis in elementary school children. *International Journal of Educational Research*, 68, 15-26. https://doi.org/10.1016/j.ijer.2014.08.001

Borko, H., Jacobs, J., & Koellner, K. (2010). Contemporary approaches to teacher professional development. *International encyclopedia of education*, 7(2), 548-556.

Bossé, M. J., Adu-Gyamfi, K., & Cheetham, M. (2011). Translations among mathematical representations: Teacher beliefs and practices. *International Journal for Mathematics Teaching & Learning, 15*(6), 1-23.

Boston, M. D., & Smith, M. S. (2011). A 'task-centric approach' to professional development: enhancing and sustaining mathematics teachers' ability to implement cognitively challenging mathematical tasks. *ZDM Mathematics Education*, *43*, 965-977. https://doi.org/10.1007/s11858-011-0353-2

Braun, V., & Clarke, V. (2013). Successful Qualitative Research: A Practical Guide for Beginners. London: Sage.

Brendefur, J. L., Thiede, K., Strother, S., Bunning, K., & Peck, D. (2013). Developing mathematical thinking: Changing teachers' knowledge and instruction. *Journal of Curriculum and Teaching*, 2(2), 62-75. http://dx.doi.org/10.5430/jct.v2n2p62

Cai, J., & Hwang, S. (2021). Teachers as redesigners of curriculum to teach mathematics through problem posing: conceptualization and initial findings of a problemposing project. *ZDM–Mathematics Education*, *53*, 1403-1416.

Charters, E. (2003). The use of think-aloud methods in qualitative research an introduction to think-aloud methods. *Brock Education Journal*, 12(2), 68-82.

Copur-Gencturk, Y., & Papakonstantinou, A. (2016). Sustainable changes in teacher practices: A longitudinal analysis of the classroom practices of high school mathematics teachers. *Journal of Mathematics Teacher Education*, 19(6), 575-594. https://doi.org/10.1007/s10857-015-9310-2

Copur-Gencturk, Y., Plowman, D., & Bai, H. (2019). Mathematics teachers' learning: Identifying key learning opportunities linked to teachers' knowledge growth. *American Educational Research Journal*, *56*(5), 1590-1628. https://doi.org/10.3102/0002831218820033

Creswell, J. W., & Miller, D. L. (2000). Determining validity in qualitative inquiry. *Theory into Practice*, *39*(3), 124-130. https://doi.org/10.1207/s15430421tip3903_2

Darling-Hammond, L., Hyler, M. E., & Gardner, M. (2017). *Effective Teacher Professional Development,* Palo Alto, CA: Learning Policy Institute.

DePiper, J. N., & Driscoll, M. (2018). Teacher knowledge and visual access to mathematics. In S. M. Uzzo, S. B. Graves, E. Shay, M. Harford, & R. Thompson (Eds.), *Pedagogical Content Knowledge in STEM* (pp. 83–102). Springer. https://doi.org/10.1007/978-3-319-97475-0_5

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher*, 38(3), 181-199. https://doi.org/10.3102/0013189X08331140

Desimone, L. M., & Garet, M. S. (2015). Best practices in teacher's professional development in the United States. *Psychology, Society and Education*, *7*, 252–263. https://doi.org/10.25115/psye.v7i3.515

Desimone, L. M., Porter, A. C., Garet, M. S., Yoon, K. S., & Birman, B. F. (2002). Effects of professional development on teachers' instruction: Results from a three-year longitudinal study. *Educational Evaluation and Policy Analysis*, 24(2), 81-112. https://doi.org/10.3102/01623737024002081

Dreher, A., & Kuntze, S. (2015). Teachers' professional knowledge and noticing: The case of multiple representations in the mathematics classroom. *Educational Studies in Mathematics*, *88*, 89-114. https://doi.org/10.1007/s10649-014-9577-8

Fauskanger, J., & Bjuland, R. (2019). Learning ambitious teaching of multiplicative properties through a cycle of enactment and investigation. *Mathematics Teacher Education and Development*, *21*(1), 125-144.

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal*, 38(3), 653-689. https://doi.org/10.3102/00028312038003653

Fuson, K., Kalchman, M., & Bransford, J. (2005). Mathematical understanding: An introduction. In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: History, mathematics, and science in the classroom* (pp. 217–256). National Academies Press.

Fuson, K. C., & Murata, A. (2007). Integrating NRC principles and the NCTM process standards to form a class learning path model that individualizes within whole-class activities. *Journal of Mathematics Education Leadership*, 10(1), 72-91.

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, 38(4), 915-945.

Heck, D. J., Plumley, C. L., Stylianou, D. A., Smith, A. A., & Moffett, G. (2019). Scaling up innovative learning in mathematics: Exploring the effect of different professional development approaches on teacher knowledge, beliefs, and instructional practice. *Educational Studies in Mathematics*, 102(3), 319-342. https://doi.org/10.1007/s10649-019-09895-6

Hill, H. C., Beisiegel, M., & Jacob, R. (2013). Professional development research: Consensus, crossroads, and challenges. *Educational Researcher*, 42(9), 476-487. https://doi.org/10.3102/0013189X13512674

Hill, H.C. & Papay, J.P. (2022). *Building better PL: How to strengthen teacher learning*. The Research Partnership for Professional Learning. https://annenberg.brown.edu/sites/default/files/rppl-building-better-pl.pdf

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal*, *105*(1), 11-30. https://doi.org/10.1086/428763

Jacob, R., Hill, H., & Corey, D. (2017). The impact of a professional development program on teachers' mathematical knowledge for teaching, instruction, and student achievement. *Journal of Research on Educational Effectiveness*, 10(2), 379-407. https://doi.org/10.1080/19345747.2016.1273411

Jacobs, J. K., Koellner, K., Seago, N., Garnier, H., & Wang, C. (2020). Professional development to support the learning and teaching of geometry: Examining the impact on teacher knowledge, instructional practice and student learning in two contexts. In P. Jenlink (Ed.), *The Language of mathematics: How the teacher's knowledge of mathematics effects instruction* (pp.143–173). Rowman and Littlefield.

Kazemi, E., Lampert, M., & Franke, M. (2009). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In R. Hunter, B. Bicknell & T. Burgess (Eds.), *Crossing divides: Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia.* (Vol. 1, pp. 12-30). MERGA.

Kennedy, M. M. (2016). How does professional development improve teaching? *Review of Educational Research*, 86(4), 945-980. https://doi.org/10.3102/0034654315626800

Koellner, K. A., Seago, N., Placa, N., Riske, A., & Carlson, D. L. (2022). Down the road: Teachers' perceptions and uptake of professional development after several years. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 43-50). PME.

Kutaka, T. S., Smith, W. M., Albano, A. D., Edwards, C. P., Ren, L., Beattie, H. L., Lewis, W.J.,Heaton, R., & Stroup, W. W. (2017). Connecting teacher professional development and student mathematics achievement: A 4-year study of an elementary mathematics specialist program. *Journal of Teacher Education*, 68(2), 140-154. https://doi.org/10.1177/0022487116687551

Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In M. K. Stein & L. Kucan (Eds.), *Instructional explanations in the discipline* (pp. 129-141). Springer.

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. *The Elementary School Journal*, 109(5), 491-509. https://doi.org/10.1086/596998

Lesh, R., Post, T., & Behr, M. (1987). Representations and translation among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33–40). Erlbaum.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematics success for all.* Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics*. Author.

Merriam, S. B. (2002). Introduction to qualitative research. *Qualitative Research in Practice: Examples for Discussion and Analysis*, 1(1), 1-17.

Neumayer DePiper, J., Louie, J., Nikula, J., Buffington, P., Tierney-Fife, P., & Driscoll, M. (2021). Promoting teacher self-efficacy for supporting English learners in mathematics: Effects of the Visual Access to Mathematics professional development. *ZDM–Mathematics Education*, 53(2), 489-502. https://doi.org/10.1007/s11858-021-01227-4

Ng, S. F., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education*, 40(3), 282-313. https://doi.org/10.5951/jresematheduc.40.3.0282

Orrill, C. H., & Brown, R. E. (2012). Making sense of double number lines in professional development: Exploring teachers' understandings of proportional relationships. *Journal of Mathematics Teacher Education*, 15(5), 381-403. https://doi.org/10.1007/s10857-012-9218-z

Orrill, C. H., Sexton, S., Lee, S.-J., & Gerde, C. (2008). Mathematics teachers' abilities to use and make sense of drawn representations. *Proceedings of the International Conference of the Learning Sciences 2008.* International Society of the Learning Sciences.

Polly, D., Neale, H., & Pugalee, D. K. (2014). How does ongoing task-focused mathematics professional development influence elementary school teachers' knowledge, beliefs and enacted pedagogies? *Early Childhood Education Journal*, *42*, 1-10. https://doi.org/10.1007/s10643-013-0585-6

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher*, 29(1), 4-15.

Santagata, R., Kersting, N., Givvin, K. B., & Stigler, J. W. (2010). Problem implementation as a lever for change: An experimental study of the effects of a professional development program on students' mathematics learning. *Journal of Research on Educational Effectiveness*, 4(1), 1-24. https://doi.org/10.1080/19345747.2010.498562

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: SAGE Publications

Stylianou, D. A. (2010). Teachers' conceptions of representation in middle school mathematics. *Journal of Mathematics Teacher Education*, *13*, 325-343. https://doi.org/10.1007/s10857-010-9143-y

Stylianou, D. A., & Silver, E. A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning*, 6(4), 353-387. https://doi.org/10.1207/s15327833mtl0604_1

Supovitz, J. A., Mayer, D. P., & Kahle, J. B. (2000). Promoting inquiry-based instructional practice: The longitudinal impact of professional development in the context of systemic reform. *Educational Policy*, *14*(3), 331-356. https://doi.org/10.1177/0895904800014003001

Taylor, J. A., Roth, K., Wilson, C. D., Stuhlsatz, M. A. M., & Tipton, E. (2017). The effect of an analysis-of-practice, video case-based, teacher professional development program on elementary students' science achievement. *Journal of Research on Educational Effectiveness*, 10(2), 241-271. http://dx.doi.org/10.1080/19345747.2016.1147628

Van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education*, 24(2), 244-276. https://doi.org/10.1016/j.tate.2006.11.005

Wæge, K., & Fauskanger, J. (2021). Teacher time outs in rehearsals: In-service teachers learning ambitious mathematics teaching practices. *Journal of Mathematics Teacher Education*, *24*(6), 563-586.

Zehetmeier, S., & Krainer, K. (2011). Ways of promoting the sustainability of mathematics teachers' professional development. *ZDM*, *43*, 875-887.

APPENDIX A

Appendix A: *Sample Codes and Exemplars*

CODE	DESCRIPTION	EXEMPLARS
Implement tasks that promote reasoning and problem solving.	Selecting tasks with multiple entry points, supporting students without taking over their thinking, encouraging students to use varied approaches and strategies	Interview excerpt: "I just wanted to capture 'what do you notice or what do you wonder' to get students to examine their work. That was from the LTG study of just getting kids before I say anything to get them to examine what the problem is asking them to do. It gives me an idea of where they are at because some kids are commenting on the fact of the difference between the numbers (on the VR) and some of them were commenting on the fact well it's sunny but it's also negative degrees. So I can get an idea how they are understanding the problem."
Facilitate meaningful mathematical discourse.	Engaging students in purposeful sharing of mathematical ideas, reasoning using varied representations, facilitating discourse among students by allowing them to explain and defend their approaches.	**Interview excerpt:** "In this clip, I'm walking around the room, facilitating this discussion and having kids talk to the room about what they're noticing. And then I'm just trying to clarify or get them to clarify what they see (in the VR) or what they're noticing. And that's something I definitely learning from the PD that I need to help guide this discussion. But it's going to be far mor rich if students share their observations and listen to the observations of their peers. Rather than me just telling them what I think they should notice.
Elicit and use evidence of student thinking.	Eliciting and gathering evidence of student understanding at strategic points, interpreting student thinking, making in the moment decisions on how to respond to student thinking	**Interview excerpt:** "So this clip fell along more of the student discussion piece of the PD rather than the content. And what i really liked, why I picked this clip is I had elicited several responses from students' previous to this. And then I came back to B. I came back to him and he had listened to several people and he had decided to change his answer. So in addition, I love that he was listening to his peers, but also he's a really thoughtful kid and he didn't just change his answer because he heard somebody say something else. He was really thoughtful about why this problem was asking him to find,"

MATH TEACHER LEARNING PATTERNS

CHARACTERIZING MATHEMATICS TEACHER LEARNING PATTERNS THROUGH COLLEGIAL CONVERSATIONS IN A COMMUNITY OF PRACTICE

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ABSTRACT

We examined secondary (6-12) mathematics teachers' participation in a professional development (PD) model where they collectively investigated video cases of students engaging with ambitious instructional materials. We leveraged frame analysis, frame processes, and the Teaching for Robust Understanding framework to characterize the learning of professional learning communities. We found that teacher learning was supported within collegial environments where teachers respectfully challenged or transformed ideas on how to solve problems of practice. Our findings highlight how engagement in a PD model supports teachers in establishing participation and reification patterns that encourage them to engage collegially, justify their positions, and align to ambitious teaching practices. These findings implicate a need for mathematics education leadership communities to take action to support collegial conversations in PD intentionally.

CHARACTERIZING MATHEMATICS TEACHER LEARNING PATTERNS THROUGH COLLEGIAL CONVERSATIONS IN A COMMUNITY OF PRACTICE

Teachers have constrained opportunities to systematically develop and share ideas about their practice (Ball et al., 2014). Even when teachers investigate teaching practice together, the mathematics education leadership community is limited in capturing their ideas so that they can be used and improved upon by others at scale (Hiebert et al., 2002). The decentralized nature of public education, coupled with the reluctance or inability to share ambitious teaching ideas, is a persistent problem and has been posited as a primary obstacle to improving American education (Charalambous & Delaney, 2020; Dewey, 1929).

This problem is important as the mathematics education leadership field continues to develop standards, assessments, and instructional materials that move teachers past lecture-based, teacher-centered instruction towards engaging students regularly in activities involving conceptual thinking, complex problem-solving, and mathematical discussions (National Council of Teachers of Mathematics [NCTM], 2014, 2018, 2020; Porter et al., 2011; Stigler & Hiebert, 1997). With this ambitious vision of mathematics instruction, there is a strong need for mathematics education leadership to provide opportunities to ground the work of teacher learning in the classroom (Gallagher, 2016; Kazemi & Hubbard, 2008). Such opportunities must also empower teachers to leverage their experiences in developing shared professional knowledge about the teaching and learning of mathematics (Hiebert & Stigler, 2017).

Professional development (PD) can be key in supporting instructional shifts that deepen learning opportunities for students (Rosli & Aliwee, 2021; Sztajn et al., 2017). Mathematics education leadership can leverage PD as a natural mechanism to empower teacher learning and contribute to a knowledge base that supports ambitious instruction. Ambitious instruction establishes "learning environments from which students emerge as agentive, knowledgeable, and resourceful thinkers and problem solvers" (Schoenfeld, 2023, p. 165). As such, this work aims to provide insight into the creation of learning spaces that can help teachers create powerful and transformative mathematics classrooms. Our work is based within a research-practice partnership that integrates key elements of coherent instructional systems within a PD model for secondary teachers. An important element of such systems is the use of ambitious instructional resources developed to support powerful mathematics teaching. Another key element of our PD model is the collective investigation of video cases featuring students engaging with these instructional resources.

In this paper, we explore how evidence of teacher learning manifests in sustained PD sessions focused on implementing mathematics instructional resources effectively. We employ a theoretical perspective of a community of practice (CoP) while incorporating principles of effective PD to understand the collective learning that occurs as these professional communities engage in both congenial and collegial dialogue. Thus, the research question guiding our work is: *How does learning about mathematics teaching practices manifest within a CoP during a PD model focused on the* *collective investigation of video cases of students engaging with ambitious instructional materials?*

Theoretical Perspective and Background

We draw on sociocultural theory to study the ways in which a community of secondary mathematics teachers engages in PD focused on ambitious mathematics teaching practices. The following sections will review the literature on the theory of learning within a CoP, PD that supports such learning, and a research-based framework that details powerful mathematics teaching practices. Furthermore, we describe the nature of congenial and collegial conversations and their relationships to teacher learning in PD settings.

Socioculturalism and Learning Within a Community of Practice

Sociocultural theorists (Brown et al., 1989; Collins et al., 1988; Lave & Wenger, 1991) argue that learning is inseparable from the activity, context, and culture in which it takes place because learning occurs through social engagement (Cobb & Yackel, 1996). Socioculturalism regards learning as participation in cultural practices and social engagements that enable learners to participate in the activities of the expert (Cobb, 1994). Furthermore, this perspective views knowing as a way of speaking and acting within cultural practices (Goos et al., 1999). According to Forman (1996), in order to facilitate learning, it is necessary to have "access to meaningful practice in a community" (p. 117) rather than focusing on instructional resources or materials (e.g., textbooks) that individual learners may use to internalize knowledge. Broadly, Lave and Wenger (1991) depict learning from this perspective as the legitimate peripheral-to-full participation in a CoP.

Communities of practice are groups of people who mutually engage in an activity, are connected by a joint enterprise, and engage with a shared repertoire of resources (Wenger, 1998a, 1998b). A CoP consists of learners, such as newcomers and more-knowledgeable others, moving from peripheral-to-full participation (Kelly, 2006; Lave & Wenger, 1991). For example, Lave and Wenger describe clothing tailors as newcomers who may learn how to cut out cloth first before learning other steps, such as sewing by hand or using a sewing machine. As the newcomers participate in a CoP of clothing tailors by learning how to perform each step of tailoring, the peripheral participation of the newcomers moves to full participation by producing a garment. In the context of teaching communities, a new teacher can enter a department as an outsider and begin by observing the normal interactions and discourse within department meetings, possibly offering passive agreement to others' discussions. Over time, this type of peripheral participation can shift towards full participation as they learn the communication norms and can authentically contribute to discussions and possibly challenge others.

Within a CoP, evidence of learning occurs through patterns of participation and reification (Wenger, 1998a, 1998b). Wenger defines participation as the experiential process of taking part in a CoP. Reification gives form to that experience through "objects that congeal this experience into 'thingness'" (1998b, p. 58). A CoP is constantly evolving in its mutual engagement among members, and the evolution of such mutual engagement can form patterns of participation indicative of the community's collective learning process. In teaching, CoPs allow members to address challenges that arise in their instructional practice by affording space to create reflective professional narratives. Professional narratives highlight practice and professional knowledge and reveal insight into cultural values (Allard et al., 2007). Because collective participation in creating professional narratives occurs through dialogue, patterns of participation in a CoP are noted as patterns that emerge in that dialogue. Participants in a CoP can create new patterns by changing how they engage in conversations within that community from one of "respectful turn taking and individual turns of talk" (Bannister, 2015, p. 357) to ones in which participants press each other for justification and ask clarifying questions in order to co-construct understanding. These changes are reified by specific community actions, including when the participants focus their discussions on a particular shared repertoire, such as a framework for best teaching practices or powerful lessons, to enhance their understanding.

Professional Development and its Design Elements

In the context of teaching and teachers, CoPs, known as professional learning communities (PLCs), can be designed and enacted by teacher leaders as an effective PD form that provides opportunities for participants to collaborate and learn. From a sociocultural perspective, PLCs are CoPs because community members are (i) mutually engaged in a communal activity of learning about and reflecting on teaching, (ii) connected by a joint enterprise to improve teaching practice, and (iii) engaged with a shared repertoire of resources, such as regular instructional routines or a common curriculum (Wenger, 1998a, 1998b). Moreover, as a PD structure, PLCs can align closely to the five elements of effective PD identified by Garet et al., (2001): content focus, active learning, coherence, sustained duration, and collective participation.

Content-focused PD grounds participants in subject matter content and focuses on how students learn that particular content (Desimone, 2011; Desimone & Garet, 2015). Content-based PD is often situated in teachers' classrooms, allowing teachers to study students' work, try new curricula, or study a particular element of pedagogy or student learning (Darling-Hammond et al., 2017). Borko et al. (2008) studied a group of teachers in a learning community focused on video cases in which all seven sessions revolved around a different mathematical task. Participants focused on aspects of the teacher's role during the enactment of mathematics tasks as well as students' mathematical reasoning with the tasks. In this PD, focused on specific mathematical content (e.g., proportional reasoning or ratios), teachers diligently worked with teacher leaders to understand the videotaped students' solution strategies, even when they did not align with any of the proposed teacher strategies. Teachers expressed that the content topics covered were meaningful, motivating the participants to learn, improve their practice, and better serve their students. Also, they found that the teachers' conversations changed to focus more on mathematical content as the PD progressed. From a CoP perspective, teachers in this PLC were able to refine their understanding of the content or shared repertoire collaboratively.

Active learning in PD refers to “opportunities for teachers to observe, receive feedback, analyze student work, or make presentations, as opposed to passively listening to lectures” (Desimone & Garet, 2015, p. 253). Active learning experiences in PD move teacher leaders away from traditional lecture modalities and instead engage teachers directly in practice connected to their classrooms and students (Darling-Hammond et al., 2017). Active learning in PD often incorporates collaboration, coaching, feedback, and modeling. It can also include analysis of student artifacts and video clips from actual mathematics classrooms. For instance, a study of PD by Alles et al. (2018) incorporated active learning by engaging teachers and teacher leaders in a learning community who worked collectively to incorporate strategies discussed in the PD into teacher planning, videotape teacher lessons, and analyze these lessons as a community. They found that teachers engaged in this PD showed a significant positive change in their dialogue practices in their classrooms compared to teachers who participated in a one-time traditional PD program. Similarly, Borko et al. (2008) incorporated active learning in their PD study of mathematics teachers through a two-year-long program utilizing the Problem-Solving Cycle model, which analyzed video from teachers’ classrooms. The active learning in this context manifested in the PD’s focus on teacher planning, implementing, and analysis of their classroom lessons. They found that, over time, teachers’ conversations became “more focused, in-depth, and analytical” (p. 432). Patterns emerged about how teachers in both of these PLCs participated and reified concepts, specifically from changes in their engagement within the PD and their teaching practices.

Coherence describes the alignment of the PD content with other aspects of a teacher’s profession. Such PD grounds teacher learning in their classroom, school, and district contexts (Kazemi & Hubbard, 2008). Thus, coherent PD content addresses teachers’ curriculum, builds on prior teacher learning, and focuses on sustained and collaborative communication with other teachers in similar contexts. Coherent PD experiences should also be relevant to teachers’ belief systems, school initiatives, and policies (Desimone, 2011; Garet et al., 2001), and support local teacher and school needs and interests (Bayar, 2014; Koellner et al., 2011). For instance, PD has been found to be more successful when coherently linked to classroom lessons. Smith et al. (2020) studied 24 teachers in a PLC from a single district that participated in PD centered on a model of professional learning in which teachers collaboratively planned and reflected on lessons they were concurrently teaching during a summer school session. The study found that the PLC members found the PD to be coherent and relevant; as a result, their practice had changed by incorporating ideas from the PD.

For PD to have sustained duration, the sessions must occur regularly over extended periods of time and remain focused on the same learning goal. Research shows that traditional, one-day PD sessions, even if there is a brief follow-up, often do not produce the intended outcomes. Ross and Bruce (2007) studied teacher learning between a group of teachers engaged in a one-day PD session with three short follow-up sessions and a control group who engaged in no PD. They found no significant difference between groups on all but one of the teacher efficacy variables and inferred the limited duration of the PD program as a way to explain this finding. Other researchers have found more sustained durations of PD to be more effective, yet the suggested duration has varied. Garet et al. (2001) suggest that teachers work together for at least one semester and have a minimum contact time of 20 hours. Yoon and colleagues (2007) found that effective PD programs averaged 49 hours of contact time. It is also important to note that more time does not guarantee more effective PD. “Time must be well organized, carefully structured, and purposefully directed” (Guskey, 2003, p. 749). For example, Santagata and Bray’s (2016) study focused on a learning community of teachers studying student mathematical errors, and illustrated how a sustained duration of PD could be designed and implemented effectively. In this PD, teachers met for two full days at the beginning of the PD and then monthly for the remaining six months of the school year. At each meeting, teachers jointly planned lessons and engaged in video analysis of teachers’ enactment of lessons. Findings indicated that the sustained duration helped the teachers grow in their understanding of students’ mathematical misconceptions and refine their practices.

Collective participation within PD refers to groups of teachers who share a common interest. PD should provide collective experiences for groups of teachers with similar needs and challenges (Desimone & Garet, 2015), such as teachers from the same grade, subject, or school. When such groups participate in PD activities together, they build an interactive learning community (Desimone, 2011), which can allow for more “collaboration, integration, and targeting of specific student needs” (Smith et al., 2020, p. 81). For example, van Es and Sherin’s (2008) study of PD with mathematics teachers illustrated the collective participation of teachers working towards the concept of noticing through mutual engagement in a video club. All participants in this study were mathematics teachers from the same district, taught similar grade levels, and were in the third year of implementing a new reform curriculum. Throughout the PD, each teacher shared video clips of their classroom activities (e.g., whole class discussion, small group work), and their peers analyzed and discussed the clips to learn to notice and interpret students’ mathematical thinking. Through the teacher leaders’ intentional design of this PD, teachers’ patterns of participation changed, wherein participants increasingly attended to detailed noticing of students’ mathematical thinking. From a CoP perspective, participation in this PD helped teachers reify the concept of professional noticing in mathematics classrooms.

As argued above, PLCs can be designed and enacted by teacher leaders as an effective form of PD that emphasizes collaborative learning and can often align closely with the five elements of effective PD proposed by Garet et al. (2001).

Threaded through all the characteristics of effective PD is the idea that teachers consistently engage in dialogue about mathematics teaching and learning. However, how teachers engage in such dialogue is also an important component impacting the effectiveness of a PD endeavor.

Congenial and Collegial Conversations

Within PD sessions, members of a CoP participate through dialogue. That dialogue generally takes the form of congenial or collegial conversation. Congenial conversations focus on politeness and privacy and are generally agreeable (Evans, 2012). Within PD sessions, congeniality could be one teacher suggesting a particular teaching move and another teacher cordially agreeing with that suggestion, regardless of their true opinion. In contrast, collegial conversations focus on constructive disagreements, development, and performance around practice (Evans, 2012). True collegiality requires more than being cordial and caring; it means examining ideas and problems of practice safely, where teachers can speak their truth without fear of repercussion (Zepeda, 2020). Within PD sessions, collegiality could be teachers disagreeing with all or some parts of their and others' suggestions for practice, which offers opportunities for members of the teacher community to suggest and argue for something different. Collegial conversations do not always mean disagreement; a collegial conversation could be one in which a community member creates a new understanding based on a posited idea.

To create a culture of growth in a PLC, teacher leaders must encourage teacher conversation that embraces collegiality because doing so authentically respects both similarities and differences (Zepeda, 2020). Collegial conversations are a catalyst for PLCs to reify their patterns of participation because these conversations allow teachers to build on or challenge each other's understanding by respecting different perspectives. In other words, collegial conversations entail deep discourse that promotes learning in a PLC. In order for communities to shift from congenial to collegial conversations, it is necessary for there to be shared repertoires for eliciting different ideas and feedback from all teachers in a PLC (Nelson et al., 2010). Collegial conversations are sociocultural because such dialogue can manifest itself as community members engage with "evolving forms of mutual engagement," "understanding and tuning their enterprise," and "developing their repertoire, styles, and discourses" (Wenger, 1998b, p. 95). Borko (2004) argues that in order to create successful learning communities, we need to create norms of interaction that support teachers to take risks in their dialogue with each other. These norms allow teachers in a PLC to discuss and justify their true opinions without the fear of dissimilar or dissenting ideas (Zepeda, 2020). In fact, recent research has shown that collegial conversations within a PLC can help teachers reify their understanding of powerful mathematics classrooms (Leonard et al., 2022).

Both congeniality and collegiality are necessary to create an effective PLC and should be actively supported by teacher leaders during PD. Congenial conversations help establish a safe space where members feel supported and their views are honored. Moreover, establishing such comfort amongst members can motivate collegial conversations, enabling the PLC to create new ideas and disagree or dissent constructively. However, not all congenial conversations lead to collegiality since the nature of congenial conversations is to avoid conflict and keep the status quo (Nelson et al., 2010; Selkrig & Keamy, 2015). Thus, in order to move from congenial to collegial conversations, the members in the PLC need to value communicative virtues, including:

> *…tolerance, patience, respect for differences, a willingness to listen, the inclination to admit that one may be mistaken, the ability to reinterpret or translate one's own concerns . . ., the self-imposition of restraint in order that others may "have a turn" to speak, and the disposition to express oneself honestly and sincerely. (Burbules & Rice, 1991, p. 411)*

Since collegial conversations involve disagreement or different opinions, as well as new meanings or honest opinions, these communicative virtues are essential to foster authentic collegiality.

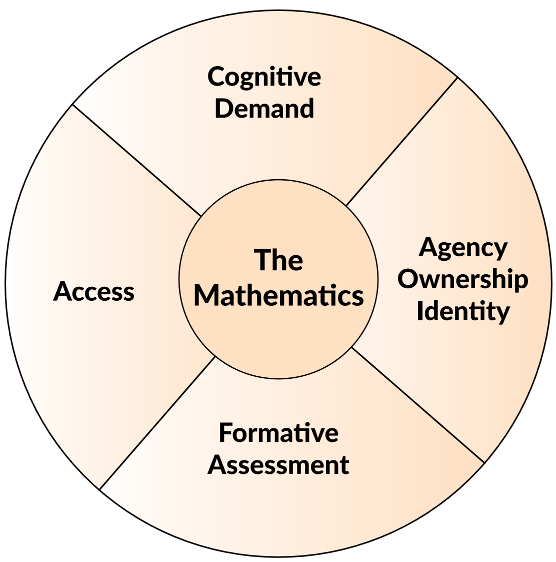
There is a caveat to the dichotomy between congenial and collegial conversations: conversations may not solely fall into congeniality or collegiality. According to Burbules and Rice (1991), different conversation forms can be categorized along the following spectrum: full agreement and consensus, partial agreement with a common understanding of different opinions, disagreement with a partial understanding of differences, disagreement with little understanding but with a respect for differences, and full disagreement without a respect for differences. This spectrum shows the complexity of conversation forms and that the classification of conversations is not absolutely dependent on the dichotomy between congenial and collegial conversations. Therefore, we interpret dialogue within PLCs as existing along a spectrum of congenial and collegial conversation. The conversation types that promote shifts from congeniality to collegiality will be discussed later in the data analysis section.

The Teaching for Robust Understanding Framework

An important aspect of a PLC comprised of mathematics teachers is the development of a shared repertoire built around best practices for teaching and learning mathematics. The Teaching for Robust Understanding (TRU) framework creates an engaging and equitable educational experience for students and aligns the PLC's vision of ambitious instruction with what occurs in powerful classrooms (Schoenfeld, 2015). The TRU framework is informed by decades of research (see Schoenfeld, 2013 for some of the history of TRU) and details five interrelated dimensions (see Figure 1 on next page): The Mathematics; Cognitive Demand (CD); Equitable Access (EA); Agency, Ownership, and Identity (AOI); and Formative Assessment (FA). When established as the focal point of a PD program, the TRU framework supports teacher learning about classroom environments in which all students are supported in becoming independent mathematical thinkers (Schoenfeld & the TRU Project, 2016).

The Mathematics
Powerful mathematics classrooms are built on rich mathematical content with which students are able to engage in meaningful ways. Such content must focus on important mathematical ideas in a coherent manner (NCTM, 2000, 2014, 2018, 2020; National Governors Association [NGA], 2010; National Research Council, 2001), reflecting the deeply connected logical structure of mathematical concepts (Schmidt et al., 2005). Nearly as important as the content students encounter in their mathematics classrooms are the mathematical practices they use to engage with that content.

Figure 1
Teaching for Robust Understanding Framework (Schoenfeld, 2017)



When students use mathematical practices, such as making conjectures and constructing mathematical arguments to justify conclusions, they actively make connections to both their prior knowledge and other ideas in mathematics (Cuoco & McCallum, 2018; NGA, 2010). Understanding that grows from this connection-making is conceptual in nature (Hiebert, 2013; Rittle-Johnson et al., 2001), and is more easily applied in novel situations (Baroody et al., 2007; Brophy, 1999; Fries et al., 2021).

Cognitive Demand
The mathematical tasks with which students engage in classrooms set boundaries for how they are able to think about mathematical content, and the depth of disciplinary understanding they are able to achieve (Doyle, 1988). Tasks that are implemented with a consistently high level of CD afford students the opportunity to struggle productively, facilitating the development of conceptual understanding (DiNapoli & Morales, Jr., 2021; Hiebert & Grouws, 2007; Warshauer, 2015). Such tasks provide improved opportunities to learn (Jackson et al., 2013; Stein et al., 1996; Tekkumru-Kisa et al., 2020), are associated with higher student achievement (Boaler & Staples, 2008; Stigler & Hiebert, 2004), and challenge students to develop sophisticated solution strategies (Downton & Sullivan, 2017). When students struggle with high-level tasks, it is critical for teachers to provide support that does not lower the CD. This can take the form of supplying adequate time for the tasks, providing proper scaffolding, and modeling effective use of mathematical practices, such as using mathematical reasoning to support a claim (Smith & Stein, 2018). Research shows that these supportive learning environments can help students persevere in their in-the-moment problem solving and nurture their willingness to productively struggle over time (DiNapoli & Miller, 2022).

Equitable Access
Access to ambitious mathematical content and instruction is important for all students and is essential to their academic and economic prospects (Moses & Cobb, 2001; NCTM, 2018, 2020). What have historically been characterized as differential outcomes in mathematical achievement associated with student gender, socioeconomic status, race, ethnicity, language, culture, and (dis)ability can more productively be framed as differential opportunities to learn (Flores, 2007; Hung et al., 2020; Milner, 2012). While many issues regarding inequitable opportunities to learn cannot be remedied at the classroom level (e.g., district-wide tracking policies), there are many ways teachers can work to provide all students access to powerful mathematics. Teachers can choose tasks that have multiple entry points and solution strategies, providing various ways students can meaningfully engage with content, thus positioning more students as capable doers of mathematics (Boaler, 2016; Hodge & Cobb, 2019; LaMar et al., 2020). Teachers can also limit their use of activities or participation structures that repeatedly privilege the same students, such as those that reward speed over depth of understanding.

Agency, Ownership, and Identity
Students' mathematical identities shape the ways in which they choose to participate in the classroom and are therefore intimately connected to their learning (Boaler, 2000; Hand & Gresalfi, 2015; Lave & Wenger, 1991). These mathematical identities are shaped by a multitude of factors, such as students' racial, ethnic, and gender identities, family and community influences, and prior mathematical experiences (Levya, 2021; Martin, 2000, 2012). Within each classroom, students' mathematical identity development is also influenced by the shared understanding of what it means to be a competent doer of mathematics in that classroom (Boaler & Greeno, 2000; Cobb et al., 2009). When teachers are mindful of students' multiple identities and position them with agency as mathematical meaning-makers, they support students in constructing positive mathematical identities for themselves (Aguirre et al., 2013). Further, when students are expected to support their ideas with mathematical reasoning and are responsible for evaluating the validity of others' reasoning, they become "authors and producers of knowledge, with ownership over it, rather than mere consumers of it" (Engle & Conant, 2002, p. 404). Teachers can support such ownership by publicly attributing ownership of ideas to students, utilizing participation structures that encourage students to build off of these ideas (e.g., think-pair-share), and by establishing classroom norms wherein mathematical reasoning and argumentation are the standard for determining the validity of student solutions,

rather than the teacher or a textbook.

Formative Assessment

Effective use of FA in the classroom has been linked to positive student learning outcomes and the development of metacognitive habits (Andersson & Palm, 2017; Black & Wiliam, 1998; Darling-Hammond et al., 2020). In contrast to summative assessment (e.g., quizzes, exams), FA is used to inform instruction rather than to evaluate student performance. FA can occur via formal classroom tasks or through in-the-moment student-teacher interactions. For example, teachers can enact pre-assessment and exit-ticket tasks to surface students' mathematical thinking. Also, teachers can ask students open-ended questions to gain insight into their thinking and understanding (Schildkamp et al., 2020), which they can then use to provide appropriate scaffolding or additional instruction. The use of FA can support students' development of a growth mindset by shifting focus away from extrinsic, performance-based motivation (Shepard, 2000), and can encourage metacognitive behaviors in students, such as self-reflection and goal setting (Granberg et al., 2021). FA pedagogies allow teachers to solicit student thinking during a lesson and adjust instruction to "respond to those ideas, by building on productive beginnings or addressing emerging misunderstandings" (Schoenfeld, 2014, p. 408), to ultimately improve teaching and learning.

Related to the TRU dimensions are Formative Assessment Lessons (FALs; see Mathematics Assessment Resource Service [MARS], 2015a). In collaboration with others, Schoenfeld's team developed FALs as instructional materials aligned to TRU. In particular, they designed FALs to be incorporated by teachers within their existing curriculum. These lessons involve tasks and activities that can foster robust, equitable learning environments where "all students are supported in becoming knowledgeable, flexible, and resourceful disciplinary thinkers" (Schoenfeld & the TRU Project, 2016, p. 3). In a study of the FALs' implementation in Kentucky, in spite of a myriad of methods that teachers chose to implement the FALs, their use was responsible for an additional 4.6 months of growth over the course of the year, based on student data from the Central for Research on Evaluation, Standards, and Student Testing mathematics assessment (Herman et al., 2015).

While each of the TRU dimensions can be viewed as distinct facets of powerful mathematics classrooms, they are all deeply connected and enhance each other as learning unfolds in the classroom. For example, providing as-needed support to all students in a way that maintains CD is heavily reliant on the in-the-moment information gathered from FA. Schoenfeld (2017) explained that "these dimensions are arranged spatially in "(Figure 1)" to illustrate both the individual dimensions and their connections – everything is connected, but each dimension has its own integrity" (p. 419). In the context of this work, the TRU framework is the core of the shared repertoire of resources for this CoP, and teachers' reification of the TRU framework includes developing an understanding of each distinct dimension as well as how they can be connected. Furthermore, the TRU framework offers a common language for dialogue within this PD setting. The next section details our methodology used to answer our research question: *How does learning about mathematics teaching practices manifest within a CoP during a PD model focused on the collective investigation of video cases of students engaging with ambitious instructional materials?*

METHODS

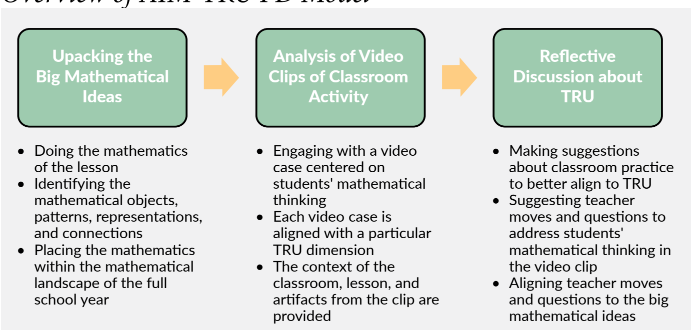
Participants and Context

This paper focuses on one of many CoPs that were part of a larger project spanning multiple regions. The CoP studied in this work consisted of three PLCs in an urban Midwestern city. The entire CoP was composed of 30 members, with each PLC containing 10 secondary mathematics teachers. Moreover, each PLC had two of its members serve as participant-facilitators. We studied this CoP for two years as they engaged in a TRU-aligned mathematics PD model called Analyzing Instruction in Mathematics using the TRU Framework (AIM-TRU). For context, most members of this CoP were from different middle schools and high schools in the region, most of which served low-income and racially diverse neighborhoods. The majority of CoP members were familiar with the TRU framework and had some experience teaching with FALs. Teachers in these PLCs had varying amounts of mathematics teaching experience, spanning 0-25 years with an average of approximately nine years. Across the two years of study, the PLCs met 24 times for 2.5-hour PD sessions conducted both in-person and via Zoom.

The AIM-TRU PD model engaged these secondary (6-12) mathematics teachers in a collaborative investigation of ambitious instructional materials to deepen instructional knowledge and support shifts in practice aligned to the TRU framework (Schoenfeld, 2015). This research team designed the model to align with Garet et al.'s (2001) five elements of effective PD: content focus, active learning, coherence, sustained duration, and collective participation. This PD model allows teachers and teacher educators to generate collective professional knowledge for teaching and learning mathematics using the dimensions of ambitious instruction that are necessary and sufficient to produce equitable environments supporting deep mathematical learning opportunities for students (Schoenfeld & the TRU Project, 2016). We have also designed our PD model in accordance with Wenger's (1998a) theory that learning occurs within CoPs, and that teacher communities can serve as levers for equitable praxis and generative settings for robust teacher learning. To leverage mathematically rich student conversations for teacher learning, the AIM-TRU PD model focuses on the following components: (a) unpacking a lesson's big mathematical ideas, (b) making observations about video cases demonstrating students' mathematical thinking while engaging in TRU-aligned FALs, and (c) sets of video case reflective discussion questions based on the TRU framework (see Figure 2 on next page). Specifically, in component (c), PLC participants were prompted to (i) posit possible teacher moves or questions that would support students in the video case to engage with the mathematics based on a particular dimension of TRU, and (ii) to align possible teacher moves or questions to the big mathematical ideas of the lesson featured in the video case. The participant-facilitators followed a detailed protocol to enact the model, which helped ensure a natural and equitable conversation among participating teachers.

Figure 2

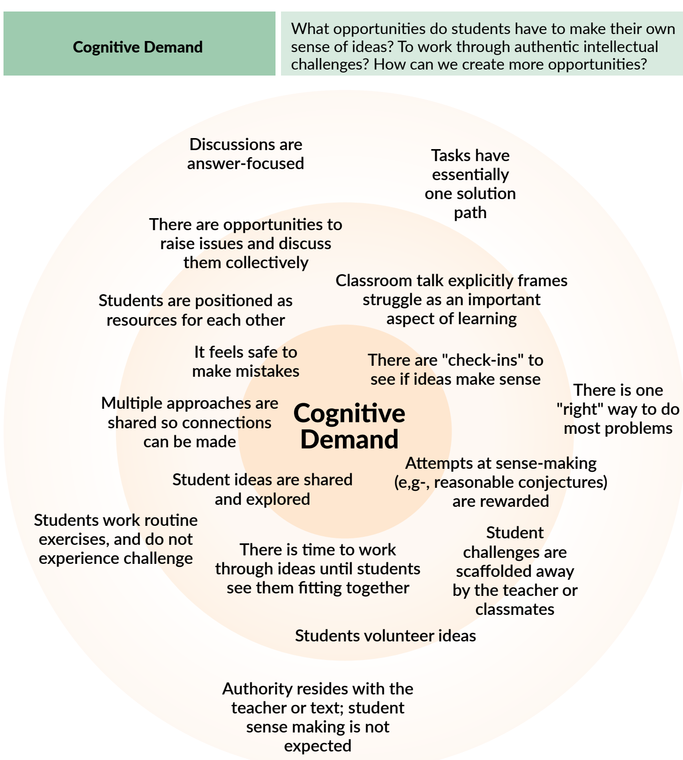
Overview of AIM-TRU PD Model



To further support participants in reflecting about the video cases relative to TRU, the AIM-TRU PD model incorporates TRU On-Target Tools (Schoenfeld et al., 2023) to situate each TRU dimension in the context of classroom activity, adapted with permission to fit our context (see Figure 3). The TRU On-Target Tools offer a visual representation of teacher moves and their alignment to a particular TRU dimension.

Figure 3

Example of a TRU On-Target Tool: Cognitive Demand



Schoenfeld and colleagues explained the On-Target Tools as follows:

> *On the outer rings of the targets are descriptions of classroom attributes and activities that are commonly found in mathematics lessons, but that, with some adjustments, hold the potential to support more equitable and ambitious learning opportunities...As you move toward the center of that target, the attributes listed describe increasingly powerful opportunities for student learning. (p. 2)*

Participant-facilitators encouraged all participants to use the TRU On-Target Tools to help them engage in reflective discussion about TRU. Thus, the TRU On-Target Tools supported participants in positing productive teacher moves that aligned to TRU and to the big mathematical ideas of the lesson featured in the video case.

Data Collection

A researcher collected video and audio recordings of the 24 PLC meetings and artifacts created by the CoP. Artifacts included shared documents capturing participants' ideas generated both individually and collectively in small group discussions during each PLC meeting. We transcribed component (c) of the AIM-TRU PD model focused on PLC participants' reflective discussion of the video cases as they related to TRU and the big mathematical ideas. We chose to focus on these reflective discussions as a data reduction strategy (see Bannister, 2015) because, in our view, those conversations contained the most concentrated evidence of teacher learning relative to our theoretical framing about how CoPs learn. All of these transcriptions were cross-referenced with the related artifacts. Thus, the primary data sources were video recordings of PLC participants studying and discussing video clips of students engaged in rich mathematical activity.

Data Analysis

To make claims about how evidence of teacher learning manifests in this context, our analysis plan considered patterns of participation and reification within the PLCs. For transparency, see Appendix A for a detailed example of the coding involved in our analysis plan. For this stage of our analysis, we focused on teacher dialogue within a particular component of the PD model that occurred after the group watched and independently reflected on the video case. After individual reflection, teachers collectively engaged with reflective discussion questions about the video case based on the TRU framework, during which they had opportunities to (i) posit possible teacher moves or questions that would support students in the video case to engage with the mathematics based on a particular dimension of TRU, and (ii) align possible teacher moves or questions to the big mathematical ideas of the lesson featured in the video case. For this section of the transcript, we applied frame analysis, a method to study the ways teachers collectively shape and structure meanings through participation and reification in a CoP (Bannister, 2015, 2018). Frames are co-constructed objects among a community that represent existing meanings in the group at any given time. Frames have been used as ways to classify and organize teacher conversations in the short term (Horn & Kane, 2015) and to demonstrate growth in a CoP over time (Bannister, 2015).

The first level of analysis was to code the transcript by core framing types: diagnostic, prognostic, or motivational (Bannister, 2015, 2018; Benford & Snow, 2000). We viewed these three frame types as different ways teachers could participate in PLC. In particular, when discussing the video case of classroom activity, a teacher could state their observation about a problem of practice (a diagnostic frame, e.g., "During group work, the students aren't listening to each other."). If the teacher provided a diagnosis, they might additionally suggest an in-the-moment teaching move that could resolve that problem of practice (a prognostic frame, e.g., "The teacher could ask one student to explain in their own words what their classmate said."). Finally, if the teacher both diagnosed and prognosed a particular problem of practice, they may also provide a rationale for a particular suggestion (a motivational frame, e.g., "Encouraging students to explain what their classmate said could help them build on each other's ideas and develop agency."). We viewed motivational frames as the most powerful type of participation within the CoP because they imply agency and motive of the community members to address the joint challenge that arose about mathematics teaching practice. Furthermore, motivational frames imply collegiality because community members are justifying a point of view that may be in contrast to earlier ideas. In previous work applying frame analysis (e.g., Bannister, 2015), researchers have used the content of the frame types to understand patterns of participation over time. In our work, we instead looked to provide additional descriptors for the frames to create a more fine-grained classification system.

Iterating on frame analysis, we recognized the need to further classify prognostic and motivational frames by their framing process to better capture the complexities of the discourse of the PLC, particularly the spectrum of conversations (Burbules & Rice, 1991) that could occur relative to congeniality and collegiality. We focused on prognostic and motivational frames because these were talk turns that contained suggested teaching moves and justifications, respectively, which aligned to our previous data reduction strategy of focusing solely on component (c) of the AIM-TRU PD model. Benford & Snow (2000) described frame processes as the several factors associated with the development of any diagnostic, prognostic, or motivational frame. Their review of the literature established several frame processes that help describe how frames are constructed in a CoP, and suggested alignment to a spectrum of conversation. The frame processes included: articulating, punctuating, bridging, amplifying, extending, transforming, countering, and disputing (see Table 1). Other than articulating, each of these frame processes implies that the central idea communicated in the frame is connected to a previous frame or frames, constructed by either building off of or contradicting others' ideas. By coding each prognostic and motivational frame according to its frame processes and by noting the transcript lines of any connected frames, we were able to capture a fuller picture of how PLC participants co-constructed ideas through dialogue (see Appendix A for a coding example). Our synthesis of frame processes and the collegiality literature revealed evidence of the alignment of certain frame processes with congenial and collegial conversations (see Table 1). We acknowledge that conversations are not binarily congenial or collegial; however, in this work, we simplified our categorization of such conversation to help us develop the general story of teachers' learning patterns as they participated in their PLCs. We viewed transforming, countering, and disputing frame processes to describe collegial discourse because these processes align more closely with Evans' (2012) and Zepeda's (2020) conception of collegiality. Specifically, these frame processes are more likely to develop new meanings, examine, and/or disagree with ideas from previous frames in ways to which others in the PLC could respond.

Table 1
Frame Alignment Processes

Category	Frame process	Definition
Congenial	Articulating	Expressing experiences, observations, and/or interpretations of implementing instructional materials
	Punctuating	Highlighting some issues, events, or beliefs as being more important than others
	Bridging	Connecting two or more unconnected frames
	Amplifying	Clarifying a previous frame
	Extending	Building on a previous frame to add insight
Collegial	Transforming	Generating new meanings or understandings based on previous frames
	Countering	Opposing or disagreeing with previous frames
	Disputing	Disagreeing with a portion of a previous frame, not the frame entirely

Since our PD model is rooted in the TRU framework, and to help us understand how the teachers reified TRU concepts, in our final level of analysis we aligned each prognostic and motivational frame with a TRU dimension and scored it using a rubric for TRU Talk in PLCs (see Appendix B). Our analysis also considered if connected frames were aligned to different TRU dimensions and if TRU alignment scores were the same or different between connected frames. This rubric is a version of the TRU Math Rubric (Schoenfeld et al., 2014), adapted with permission from Dr. Schoenfeld to fit our context and in collaboration with our project's external evaluator. This rubric partitioned each dimension of TRU into three numeric levels, with level 3 being the highest rating for teacher talk aligned with powerful mathematics classroom activity. When a frame did not clearly align with whole number scores, half-scores were assigned.

Bannister (2015) focused on individual members of a CoP by analyzing changes within prognostic frames related to pedagogical strategies generated by specific teachers. Incorporating frame processes and TRU alignment of proposed teaching moves allowed us to leverage Bannister's model to analyze the CoP as a whole, rather than individual teachers. By analyzing changes in frame processes and TRU alignment scores, we were able to capture patterns of participation and reification by noting how the dialogue as well as PLC ideas evolved within PD sessions.

RESULTS

The results reported here are informed by our analysis of teachers' participation in component (c) of the AIM-TRU PD model focused on PLC participants' reflective discussion of the video cases as they related to TRU and the big mathematical ideas of the lesson. This section addresses the research question that guided this study, namely how teacher learning manifested within a CoP situated in the AIM-TRU PD model. We answer this research question in two ways. First, we articulate our general findings about the ways in which teachers in their PLCs participated in the AIM-TRU PD model across all sessions in the full two-year data set. Second, we illustrate how teachers in their PLCs changed the ways they participated within sessions with descriptions of representative excerpts from our data set.

General Findings: Teachers' Participation in Their PLCs

Over the course of the three PLCs, we coded 226 frames in which an individual teacher participant offered a prognosis for a problem of practice observed in the video case of classroom activity, or a motivation for such a prognosis. These frames occurred as a part of natural conversations among colleagues, and all 30 teacher participants are represented in these frames. Our analysis of this dialogue revealed that teachers participated both congenially and collegially in their PLCs. They also participated by motivating their prognoses and leveraging TRU concepts during conversation. In general, we found that when teachers participated in collegial frame processes, they engaged more often in motivational frames and their conversations aligned more closely to the TRU framework. Furthermore, we found that these types of participation connected to reification of the TRU dimensions, as evidenced by higher TRU scores during collegial dialogue and more connections made between multiple TRU dimensions when compared to congenial dialogue (see Table 2). Teachers engaged most often in congenial conversation, with 77% of teachers' prognostic or motivational frames (174 total frames) being classified as an articulating, punctuating, bridging, amplifying, or extending frame process. At other times, teachers participated in collegial conversation, with 23% of teachers' prognostic or motivational frames (52 total frames) being classified as a transforming, disputing, or countering frame process. This finding shows that teachers engage in dialogue in various ways within their PLCs and indicates that understanding participation and reification through the lens of congeniality and collegiality can provide important information about the nature of their learning.

Table 2
Summary of Teachers' Participation in PLCs

Frame characteristics	Congenial frame processes	Collegial frame processes
Prognostic and Motivational Frames (%)	174 (77%)	52 (23%)
Motivational Frames (%)	54 (31%)	33 (62%)
Average TRU Score	2.39	2.82
TRU Dimension Change (%)	14 (8%)	21 (40%)

Within teachers' congenial and collegial participation, we found two ways teachers participated in the AIM-TRU PD model that were impactful to their learning: by engaging in prognostic or motivational frame types, and by the ways they aligned their frames to TRU dimensions. By proposing an in-the-moment instructional solution or providing an accompanying rationale for an in-the-moment instructional solution, teachers in the PLCs toggled between prognostic and motivational frames, respectively, as a method of sharing their suggestions for teacher moves. When conversations were congenial, teachers' frames were motivational 31% of the time, which means that teachers' frames were prognostic and did not offer a motivation for a proposed solution the other 69% of the time. In contrast, when conversations were collegial, teachers motivated their proposed teaching moves and connections to the big mathematical ideas 62% of the time, which means that teachers' frames were prognostic the other 38% of the time. This finding shows that teachers participating in a collegial environment were doubly likely to justify their prognosis to their peers, compared to when participating in a congenial environment, which suggests the teachers were able to leverage the collegial environment to participate within the CoP more powerfully with agency and motive. Additionally, teachers in the PLCs varied the degree to which their frames aligned to TRU dimensions. When conversations were congenial, teachers' frames were assessed to have an average TRU alignment score of 2.39 out of 3; 8% of these frames were connected to a previous frame and involved a change in TRU dimension alignment (see our Representative Excerpts below for examples of this), which shows an understanding of the interrelatedness of the TRU framework. Alternatively, when conversations were collegial, teachers' frames were assessed to have an average TRU alignment score of 2.82 out of 3; 40% of these frames were connected to previous frames and involved a change in TRU dimension alignment. These findings show that teachers participating in a collegial environment were positing teacher moves that were more closely indicative of ambitious mathematics instruction, compared to when participating in a congenial environment. These findings also show how teachers' reification of TRU concepts, via higher TRU alignment scores and more emphasis on making connections between TRU dimensions, were more prevalent in collegial environments.

Overall, these general findings suggest that when teachers engage in the AIM-TRU PD model, specifically in component (c), collegial dialogue promotes participation in the form of motivational framing and alignment to TRU dimensions. This participation type supports teachers in reifying TRU dimensions and how the dimensions relate to the big mathematical ideas of the lesson. The next section presents representative excerpts from the AIM-TRU PD model sessions. These excerpts will help show ways in which teachers changed their participation within sessions and help us understand teacher learning within these PLCs.

Representative Excerpts: Illustrating the Changes in Teacher Participation in PLCs

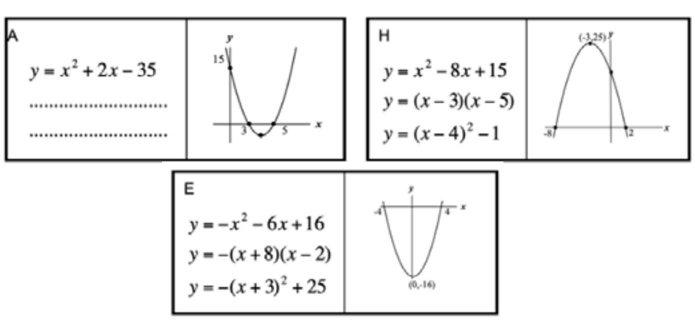
The general findings indicate that collegial frame processes appear to have stronger TRU alignment scores and more connections between TRU dimensions, therefore showing evidence of reification of TRU concepts within PLCs. We present three representative excerpts illustrating specific instances of teachers from various communities changing their participation to help describe how teachers' learning patterns may have emerged. Each of the three excerpts below provides a window into a frame process aligned with collegial conversations: countering, transforming, and disputing, respectively. The excerpts were chosen to provide examples from each of the PLCs within the CoP, and to illustrate different frame processes in context. All of these representative excerpts illustrate the duality of changes in participation and reification within PD sessions, and thus, illustrate how teacher learning manifested from a CoP perspective. To help the reader recognize the different frame processes in these excerpts, we highlighted the relevant text in the transcript and in the corresponding analysis that follows according to the color scheme in the Frame Alignment Processes showcased in Table 1.

Excerpt I: Countering and CD in the Context of Quadratic Functions

During the first year of the PD model, there were a number of congenial frame processes, but during the sixth session we found evidence that this community of mixed middle and high school mathematics teachers changed their participation to shift into collegial dialogue. Here, the PLC was investigating a video case centered on representing quadratic functions graphically. Teachers discussed whether students were struggling unproductively with a domino lesson activity (Figure 4) in which they created links between quadratic graphs and their algebraic representation.

Figure 4

Example Cards from Activity in PD Session 6 (MARS, 2015c)



In the following transcript[1], Teachers 1, 2, and 3 prognose teaching moves related to the organization and presentation of the task. Teacher 4 then questions the need for such alterations and prognoses a teaching move:

Teacher 1: I also looked at one of the ideas on the outside of the target: "discussions are answer-focused." So, the students were definitely praising their struggle and being like, okay, progress at the end. But it kind of seemed like they were still like, "I gotta do all these things, and oh my gosh, there's so many cards." I wonder how this task would have changed or their approach would have changed if we just gave them three cards to look at or three totally random cards, you don't even need to connect. But look at these and see what you make of them.

Teacher 2: Even if instead of giving them both sides of the card, maybe just splitting them and giving them just one portion and seeing what they would do with it. Seeing if they could, for certain cards, seeing if you gave them the graph, that they come up with the equation, that they come up with the factored format, that they come up with, just whatever they can pull from it. Then if you gave them a side that had some of the equations there or maybe one of them, could they come up with the other pieces, to kind of see how much they know and understand, and how it can interrelate before they get the piece with the picture.

Teacher 3: The FAL, it recommends that you start the kids out with just two cards, A and H, and you give them or you give them like three, I think. And you just give them these three cards and then they talk about matching them and how it works and stuff. So I think . . . if you follow the lesson structure, it sets up the kids, we'll look at one at a time, instead of going all over a little bit.

Teacher 4: I totally understand. But, [Teacher 5], I really enjoy watching your class, I thought that they did a phenomenal job even through the productive struggle. But even when we look back at that cognitive demand bullseye, they could very well have started them with three or even just two. But then working through that and pushing through and making reference to their previous notes really shows that they were, they had some type of knowledge on how to maneuver that y-intercept or that x-intercept and substituting it for different numbers. I just thought that if we had more

1. All transcripts in this paper have been edited to include gender-neutral pronouns.

time... I think that we would have seen an even more successful lesson where the kids would have been able to do that. I think just them having that mindset of even pushing through, I thought it was a really good job. I understand starting with two or three, but they were not giving up and they were making reference to their notes, and whether the notes were in a notebook or on the walls, they knew exactly where to go to find those answers or something that will lead them to an answer.

In this interaction, Teacher 1 articulated a prognosis that the teacher in the video could have reduced the initial number of dominos to combat students feeling overwhelmed by the number of cards within the task. This initial frame from Teacher 1 was coded as Level-2 TRU alignment to CD because Teacher 1 suggested a move that could help keep students productively engaged with central mathematical ideas but scaffolded away some of the challenge. Teacher 2 responded in a second frame with a prognosis to change the activity to a matching activity to help students focus on making a single link between a quadratic graph and equation, rather than making several links across representations. This was an example of a transforming frame process because Teacher 2's new prognosis transformed Teacher 1's original prognosis to focus on a new activity to help keep students productively engaged. This frame was also coded as a Level-2 TRU alignment to CD because Teacher 2 suggested a move that could help keep students productively engaged with mathematics, but also scaffolded away some of the challenge. While Teacher 2's frame is a collegial frame (transforming), the prognosis did not strengthen the TRU alignment score or shift the TRU dimension of focus. Teacher 3 then reminded the community that their suggestions are actually part of the FAL's directions. This frame was coded as an amplifying frame connected to Teacher 1's original prognosis. This frame was coded with a Level-2 TRU alignment score because Teacher 3 did not alter the original prognosis, but rather supported the suggestion by clarifying that the teacher move is embedded in the directions for this FAL.

The final frame in this example was provided by Teacher 4 when they pushed the collegial conversation further by countering suggestions made by the three previous teachers. Teacher 4 disagreed with the prior diagnosis that students were struggling unproductively in the classroom video and asserted that teacher intervention was not needed to help students productively struggle. Instead, Teacher 4 prognosed that the teacher could provide the students more time to continue to engage in the mathematical practices that they were using when faced with uncertainty, such as referencing prior resources and displaying the mindset to grapple with the content. Teacher 4 believed that these practices were aligned to a high level of CD, and by providing them with more time, the teacher would see the students successfully navigate the task and make important connections. This countering frame was coded as a Level-3 TRU alignment to CD because the suggestion requires students to continue to engage in mathematical practices without scaffolding away the challenges by providing students adequate time to struggle with the core content. The increase in TRU alignment score indicated reification of the CD dimension within the collegial dialogue.

Excerpt II: Transforming from FA to AOI in the Context of Properties of Exponents

During the third session of the second year in a PLC of middle school mathematics teachers, we found evidence of changing participation and reification via a transforming frame that built on two previous frames, shifted the TRU dimension of focus, and increased the TRU alignment score. The PLC was investigating a video of three students completing a card sort with exponential expressions. In the video, Student 3 relied on a calculator to match equivalent cards and did not respond to suggestions made by two other students (Student 4 and Student 5), who applied exponent rules to match cards. To make a match for the card $6^8 \div 6^4$, Student 4 and Student 5 told Student 3 several times that because the bases were the same, the exponents could be subtracted. Student 3 insisted on evaluating the expression on a calculator first, writing out 1,679,616 ÷ 1,296, and again used a calculator to find this quotient before choosing an equivalent card.

In the following discussion, Teacher 6 suggests a prognosis which is amplified by Teacher 7 before a teacher facilitator poses a question to the community. Teacher 8 then transforms the previous frames:

Teacher 6: When they were writing out 6^8 and they wrote out the big number, as a teacher, what would I say is, is there another way we can represent that 6^8? To help them see and then connect between what Students 4 and 5 were talking about. And what Student 3 was, how they were interpreting that number and then see how it would play out with the division.

Teacher 7: Yeah, I think that piece right there [referring to target] was very powerful, that "Tasks have multiple entry points." Students 4 and 5, I don't know if they understand or memorized the properties that the teacher taught. And Student 3 was able to use a computational [approach]. As a teacher, we could have walked in, and try to get them to make that connection, like that's great what you're doing, Student 3, but what if you don't have a calculator? What can you do to solve this problem for those moments, maybe, and hopefully tie in what Student 4 and Student 5 were thinking? Piece it together to help them make that connection.

Facilitator 1: Just thinking about what even, the comments that we just heard, and even what [Teacher 7] just said about making the connection with evaluating. How can we use all of that to help tie in what that overall big idea should be, and even looking at the notes and things that we've jotted down throughout the session? How can we bring that together?

Teacher 8: I just wanted to add on there, on that [target] where it says, "Students have opportunities to explain." They do, and they were making a claim, but Students 4 and 5 weren't following it up with any evidence or reasoning. Maybe having something there for them as a reminder. When you're working in the group that, because they just kept repeating, "because it's the same base, same base!"

During the discussion of possible teaching moves, Teacher 6 articulated a prognosis that the teacher in the video could have asked Student 3 if there was another way to represent 6^8, guiding them to think about the expression using exponent rules rather than using a calculator. This initial frame was coded as an articulating frame because Teacher 6 presented a new prognosis unrelated to previously discussed teaching moves in this session. Teacher 7 then suggested that the teacher could have asked Student 3 how they would make a match if they did not have a calculator. This is an example of a congenial, punctuating frame as Teacher 7 is restating Teacher 6's prognosis, highlighting the need to shift Student 3's reasoning away from the calculator without changing the original prognosed teaching move. These prognoses were both coded as FA Level-2 TRU alignment score because the suggested questioning would elicit student thinking but plans to build on the student's ideas were not articulated.

The teacher facilitator then probed the community of teachers to think more deeply about their prognoses and to make connections to their generated big mathematical ideas for this lesson. Teacher 8 then responded with a transforming frame by suggesting that providing students with something to remind them to justify their mathematical claims might have helped Student 4 and Student 5 expand their mathematical explanations beyond just pointing out that both 6^8 and 6^4 have the same base. Teacher 8 then motivated their prognosis:

Teacher 8: If they would have followed it up, just, and shown [Student 3] why it works, that would have maybe helped, or got them thinking on a different strategy.

This is an example of a collegial, transforming, motivational frame because Teacher 8's prognosis sought to address Student 3's over-reliance on the calculator by shifting the focus from the teacher questioning suggested by Teachers 6 and 7 (FA) to encouraging students to take responsibility for explaining concepts to their peers (AOI). This is an example of a change in participation because this was the first time in this exchange that a teacher provided a justification for their prognosis. Additionally, the shift in TRU dimension is an example of how teachers in this community used transforming frames to change the TRU dimension under investigation. This frame was coded with a higher Level-2.5 TRU alignment score in AOI because the suggested move would facilitate students coming to an agreement without the teacher acting as the arbiter of correctness. The shift in TRU alignment and the greater TRU score indicated reification of the interrelatedness of the TRU framework as well as both the FA and AOI dimensions within the collegial dialogue.

Excerpt III: Disputing and EA in the Context of Linear and Exponential Growth

During the fourth session in the second year, we found evidence of changing participation and reification via a disputing frame in the community of high school teachers. This representative example occurs after the community had watched a video of students completing the first card sort of an FAL about representations of linear and exponential growth. In this card sort, students need to match investment plans to formulas that model each plan (Figure 5).

Figure 5

Example Cards from Activity in PD Session 4 (MARS, 2015b)

P5 Investment: $400 Compound Interest Rate: 8%	**F1** $A = 400 \times 1.08^n$
P6 Investment: $400 Simple Interest Rate: 2%	**F4** $A = 400 + 8n$

The community of teachers is reflecting on the student interactions in the video through the lens of EA. Prior to the excerpt from the conversation, the community discussed that one of the three students does not appear to be participating in the small group discussion. The teachers prognosed multiple teaching moves to address the inequitable participation: holding a conference with students to discuss the exclusion of one student, establishing checkpoint protocols before moving to another card, probing student thinking about what they heard the group say, developing student-to-student questions as a standard practice in the class, and reminding students of class participation expectations. The excerpt below begins with additional prognoses, then transitions into one teacher disputing the general understanding of the community:

Facilitator 1: I just think from an equity point of view. This is not just access, but it's equitable access. If we're letting some kids not participate and we're letting other kids not let them participate. Are there other moves you all can think of that in terms of equitable access you do to try and prevent this kind of thing?

Teacher 9: I used to do this one activity, where students, even in a group, each student would have a different question but relating to the same topic, regardless of what we were studying. So each student had to come up with an answer and their own process first, and then they would compare … Then they would switch questions with other groups. In the end, we were able to have a class discussion based on the same questions, but each student was responsible for one within each set.

Teacher 10: I've done it before, where we've had a group working, and they each have a different role, and then they rotate. One person might be in charge of the explanation, another person would be in charge of recording it, and the other person will be presenting it. Depending on which one they had, they had to be prepared for their own thing… So for the student who might not have been able to develop it, at least they would have to have the understanding of how to explain it if they were chosen to present that...

Facilitator 2: Yeah, along those lines, just go to Student 3, and be like, "Hey Student 3, I'd like you to be the one to write on this blank card." And then walk away. Easy way to increase the equitable access in the moment.

Teacher 10: It's important when you're looking at the group ... is the focus on completing the task? Or making sure that all the people in the task are involved and understand all of the steps? So it doesn't have to be completed if it can be demonstrated that everyone had a say in it and took part in it. Sometimes the difficulty for the students is making sure that they can explain it in a way that somebody else understands it. Not that they can demonstrate that they themselves understand. So instead of being the knowledge of the task, the communication of what they're doing might be the focus of the activity for them.

During this interaction, Teacher 9 continued to address the issue of uneven participation among students by suggesting the teacher provide each student a similar, but varied set of problems to give the students a chance to discuss the similarities with mathematical processes. This articulating frame was coded as a Level-2 for EA because the teacher is attempting to develop a structure for equitable participation structures but does not detail how this move could achieve meaningful participation from all students in the group. Teacher 10 then articulated a new prognosis to assign roles for each student: record, explain, and present the group's mathematical thinking. This prognosis was coded as a Level-2.5 for EA because while Teacher 10 provided a teacher move that could achieve broad participation, not all of the student roles can be considered meaningful participation with the mathematical content. For example, a student assigned the role of recorder can passively take notes and not engage with core mathematical practices. Facilitator 2 provided a punctuating frame for Teacher 10's articulation when they suggested that the teacher have the non-participating student be the one to write the equation down. As a set of frames, these talk turns are an example of a congenial conversation. Teachers and facilitators alike articulated new prognoses, politely agreed with each other, and did not challenge each other's thinking.

The general consensus to this point in the discussion was that teacher intervention was needed to have one student participate in group discussions. In Teacher 10's next frame, there is evidence of a disputing frame when they differ from their own previous prognosis as well as those prognoses that came previously by offering a new perspective on the video clip they watched. Each of the previous teacher moves was centered on having all students discuss the outcome and finished product of the card sort and the task. In the final frame presented, Teacher 10 proposed the teacher shift the focus from the completion of the card sort and task to the creation of a learning goal related to group understanding through communication practices. Teacher 10 then motivated their prognosis by claiming that changing the goal of the group to making sure everyone in the group understands the math might encourage students who might not otherwise participate to share their ideas. This disputing frame was coded with a Level-3 for EA because it was a detailed, specific teaching move that has the potential to achieve and support meaningful participation within the group. This new disputing frame transitioned the conversation to a collegial conversation and also increased the TRU alignment score as the disputing frame is connected to previous frames, thus indicating reification of the EA dimension within the collegial dialogue.

DISCUSSION

Our analysis of these PLCs revealed the ways in which teachers participated in the AIM-TRU PD model, the specific participation types that supported reification of TRU concepts, and evidence of changes in participation and reification from teachers within PD sessions. These findings also imply actionable facilitation practices that could inform how teacher leaders support teacher learning within mathematics PD. Iterating on Bannister (2015, 2018), we found evidence of different types of participation patterns through identifying teachers' frames within PD sessions: when the conversation consisted of collegial frame processes, teachers were more likely to engage in motivational frames and TRU-aligned suggestions about teaching moves. These TRU-aligned suggestions also provided evidence of reification of teaching and learning across the five dimensions through connections teachers made from one dimension to another, illustrating the dimensions' interrelatedness (Schoenfeld & the TRU Project, 2016). From a CoP theoretical perspective (Wenger, 1998a, 1998b), these patterns in the nature of the dialogue within PD sessions help illustrate evidence of teacher learning because members of the PLCs demonstrated the duality of participation and reification. Specifically, as members of the PLCs established patterns of participation conducive to collegiality and motivational framing, these styles of discourse were indicative of their learning process about TRU-aligned teaching.

When analyzing the PLC dialogue within PD sessions, teacher learning patterns were most clear when teachers transitioned from congenial to collegial conversation. For instance, we saw evidence of this in all excerpts, but particularly in Representative Excerpt I as Teacher 4 leveraged the TRU On-Target Tool to counter their peers' earlier prognoses and suggest a teacher move more supportive of students' productive struggle with graphs of quadratic functions and their algebraic representations. These instances are indicative of Borko's (2004) and Zepeda's (2020) successful learning communities for teachers, as teachers in their PLCs felt safe to take risks in their dialogue by respectfully challenging each other. It is notable that Teacher 4 prefaced their countering prognosis by referencing the TRU framework (i.e., the CoP's shared repertoire) via the TRU On-Target Tool. Related to Nelson et al.'s (2010) position on leveraging shared repertoires to help elicit collegial ideas and feedback within a PLC, couching a countering frame within the TRU framework made it easier for Teacher 4 to challenge their peers because the teachers' perception of Teacher 4's countering prognosis was not personal, instead it was aligned to TRU concepts. Facilitators can direct participants' attention toward the CoP's shared repertoire, which can provide participants a safe way to engage with each other collegially. In this way, skilled facilitators are imperative for helping shift the PD dialogue from congenial conversation that builds trust to collegial conversation that can create new ideas through constructive disagreement (Burbules & Rice, 1991).

Teacher learning patterns in dialogue were also apparent via collegial, motivational frames, as teachers began to offer rationalizations to their prognoses for a problem of practice. For instance, we saw evidence of this in Representative Excerpts II and III. Specifically, in Excerpt III, we see Teacher 10 justifying their idea to shift the focus from the completion of the card sort task to more student-to-student discussion about the meanings of linear and exponential growth because it encourages all students to participate in sharing their thinking. This motivational frame occurred while Teacher 10 collegially disputed earlier prognoses made by others in the PLC, as well as self-disputing their own previous prognosis. Not only do these types of instances highlight the importance of collegiality, but they also highlight the importance of teachers sharing their motivations for their ideas (Benford & Snow, 2000) about instructional practice within PD. Relevant to teacher leaders, PD facilitators should establish norms during sessions that encourage justification of any and all ideas, perhaps especially ideas that are in discord with others. Finally, these patterns inform teacher leaders about how design elements of PD programs, such as reflecting on ambitious teaching practices via video case analysis (Alles et al., 2018; Borko et al., 2008; Garet et al., 2001), can support such motivations to be shared.

Lastly, teacher learning patterns through dialogue were evident when teachers began to use their suggestions for instruction to make connections between the dimensions of the TRU framework. For instance, in Representative Excerpt II we see teachers suggesting instructional moves aligned with both FA and AOI. Particularly, we see Teacher 8 shifting the focus from FA to AOI by suggesting that the teacher remind students in the video clip to take responsibility for explaining concepts of exponential properties to their peers. Such student-to-student discourse could cultivate new understandings without the teacher acting as the arbiter of correctness. This collegial, transforming, and motivational frame aligned the teacher dialogue more closely to the TRU framework and showcased possible connections between FA and AOI. Following Facilitator 1's prompting to focus on the PLC's big mathematical idea, Teacher 8's prognosis also refocused the discussion on the lesson content, i.e., properties of exponents. These instances underline the importance of collegiality and skilled facilitation within content-focused PD (Darling-Hammond et al., 2017; Garet et al., 2001). Mathematics-content PD can be a challenging place for teachers because they may fear judgment about their content knowledge and withhold their full participation. Encouraging collegiality is especially important in these settings because PD should stimulate discourse about the mathematics content itself, in addition to the pedagogy, to discuss and open up opportunities for growth in content knowledge. Moreover, collegiality is critical for any PD grounded in the TRU framework because the mathematics content is at the center of TRU, and without a deep understanding of the mathematics, no authentic learning can be realized across the other pedagogical dimensions (Schoenfeld & the TRU Project, 2016). Teacher leaders can support such learning within PD by challenging participants to focus their comments on the mathematical content that is the focus of the PD session.

These findings also implicitly contribute to teachers' identity development within a CoP. Lave and Wenger (1991) argue that "learning and a sense of identity are inseparable: they are aspects of the same phenomenon" (p. 115). Although our research question was not directly focused on teacher identity, another way to interpret the duality of participation and reification we found in our study is to view it as evidence of the development of teachers' relationships between themselves and their place of membership in their PLC. Teachers in these PLCs developed their identities as effective mathematics practitioners, as evident by their negotiation of different points of view about how to solve problems of practice and how those solutions align with TRU concepts. As we have alluded to in the previous paragraphs, our findings implicate action for members of the mathematics education leadership community, namely via the design of PD models. The AIM-TRU PD model studied here was intentionally aligned to the design elements of Garet and colleagues (2001), particularly to focus on rich mathematics content, active learning, coherence, sustained duration, and collective participation. We quickly learned, however, that collegial conversation within PD activities is vital to support teacher learning about ambitious mathematics instruction. PD models need to be intentional about how to cultivate collegial environments and invite productive disagreement aligned closely to students' opportunities to learn rich mathematical content. Furthermore, these findings implicate action for facilitators of such PD models to create opportunities for dialogue to transition from congenial to collegial. Within our larger project, we have been reflecting on these findings and intentionally revising the AIM-TRU PD model, specifically through our facilitation guides. The goal of these guides is to equip facilitators with questions that invite more collegial dialogue among members of the PLCs. It is important for mathematics education leadership groups to find ways to support facilitators in this way. Many researchers have shown the critical role facilitators play in supporting and fostering productive teacher learning (e.g., Borko et al., 2021; Coles, 2013; Lesseig et al., 2017), yet there is a lack of research available on how to support teacher leaders in facilitating PD with their peers.

LIMITATIONS AND FUTURE RESEARCH

This work had some limitations that will inform future research. First, we were only able to make claims about PLCs' changes in participation and reification within PD sessions, not generally across all PD sessions. This is because our scope of analysis for this paper did not consider the specific ways in which PLCs evolved in collegiality over time. Therefore, we cannot presently make a claim about how all the PLCs changed their participation from congeniality to collegiality over the course of the two years. Instead, we focused on making claims within PD sessions, to better understand how collegiality manifested in our context and how it might be useful to help teachers reify TRU concepts. Future research will consider the evolution of collegiality across all PD sessions, and thus, more general statements about the duality of changes in participation and reification in all PLCs.

Related to this point, we used a binary framework of congenial and collegial frame alignment processes (Benford & Snow, 2000), which required us to describe PLC conversations as one or the other. This allowed us to tell only a binary story of the dialogue. In reality, frame process types can vary in congeniality and collegiality (e.g., punctuating frame processes may be more congenial than extending frame processes; disputing frame processes may be more collegial than transforming frame processes). Because our research question was exploratory, we made the decision to binarily consider frame processes that were either congenial or collegial to help us understand how learning was manifesting in PLCs. Future research will consider a finer grain size of congenial and collegial dialogue and contribute to the field by defining and operationalizing a spectrum of congenial and collegial conversation, inspired by Burbules and Rice's (1991) work on the plethora of communicative virtues.

Another limitation of this work is that our research question was not focused on the impact of certain facilitation moves to support teacher learning. Although our analysis of teachers' participation in a CoP helped us infer ideas about productive facilitation, we did not study this directly. It is imperative for the mathematics education leadership community that future research investigate the relationship between facilitation moves and teacher learning, specifically how facilitation moves can support collegial conversations during PD.

In addition to future work motivated by the stated limitations, we plan to conduct larger-scale studies to continue our research. The focus of the current research question did not warrant conducting statistical analyses to show significant differences in our findings. This project is ongoing, and we continue to iterate on this work with the goal of testing a larger sample size of frames for significant differences in the occurrences of collegial frames and their associated TRU scores. Also, we only considered one regional site. This paper is part of a larger project that studies PD in several regions, all with unique settings and needs, and future studies within this project will consider all regions to help make claims about how teacher learning can manifest in different contexts.

Furthermore, the current research question allowed us to solely focus on the evidence of teacher learning that manifested within one component of the AIM-TRU PD model: the (c) sets of video case reflective discussion questions based on the TRU framework. Future research within this project will expand the scope of focus to include how teachers' duality of participation and reification manifests in the model's other components, (a) unpacking a lesson's big mathematical ideas and (b) making observations about video cases demonstrating students engaged in rich mathematical activity, and how those experiences might influence the ways in which teachers posit potential solutions to problems of practice in mathematics classrooms.

CONCLUSION

The goal of this research was to investigate teacher learning within CoPs focused on ambitious mathematics instruction. We found that teacher learning is supported within a collegial environment where teachers can respectfully disagree on how to solve problems of practice in mathematics classrooms. Our findings highlight how engagement within a PD model can support teachers to change their participation and reification patterns to more often engage collegially, justify their positions, and align their positions to research-based frameworks aimed at ambitious teaching practices. These findings allow us to respond to national calls for PD to center on teacher dialogue about classroom practices and to construct new ideas about mathematics teaching and learning (Hiebert & Stigler, 2017; Kazemi & Hubbard, 2008). Our analytic use of frame processes (Benford & Snow, 2000) and TRU Framework alignment (Schoenfeld et al., 2014) extended the frame analysis work of Bannister (2015, 2018) and afforded us the opportunity to identify three distinct manifestations of collective teacher learning within PD sessions: advancing collegiality, increased motivational framing, and alignment of conversation to TRU concepts. At large, future research in mathematics education leadership should focus on how to intentionally foster collegial interaction in PD to support teacher learning, through facilitation support and design elements, as well as examining how teachers' participation in collegial PD models impacts their actual classroom practice. ■

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REFERENCES

Aguirre, J., Mayfield-Ingram, K., & Martin, D. B. (2013). *The impact of identity in K-8 mathematics learning and teaching: Rethinking equity-based practices*. National Council of Teachers of Mathematics.

Allard, C. C., Goldblatt, P. F., Kemball, J. I., Kendrick, S. A., Millen, K. J., & Smith, D. M. (2007). Becoming a reflective community of practice. *Reflective Practice, 8*(3), 299-314.

Alles, M., Seidel, T., & Gröschner, A. (2018). Toward better goal clarity in instruction: How focus on content, social exchange and active learning supports teachers in improving dialogic teaching practices. *International Education Studies, 11*(1), 11-24.

Andersson, C., & Palm, T. (2017). The impact of formative assessment on student achievement: A study of the effects of changes to classroom practice after a comprehensive professional development programme. *Learning and Instruction*, *49*, 92-102.

Ball, D. L., Ben-Peretz, M., & Cohen, R. B. (2014). Records of practice and the development of collective professional knowledge. *British Journal of Educational Studies, 62*(3), 317-335.

Bannister, N. A. (2015). Reframing practice: Teacher learning through interactions in a collaborative group. *Journal of the Learning Sciences, 24*(3), 347-372.

Bannister, N. A. (2018). Research commentary: Theorizing collaborative mathematics teacher learning in communities of practice. *Journal for Research in Mathematics Education, 49*(2), 125-139.

Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). Research commentary: An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education*, *38*(2), 115-131.

Bayar, A. (2014). The components of effective professional development activities in terms of teachers' perspective. *International Online Journal of Educational Sciences*, *6*(2), 319-327.

Benford, R. D., & Snow, D. A. (2000). Framing processes and social movements: An overview and assessment. *Annual Review of Sociology, 26*(1), 611-639.

Black, P., & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education: Principles, Policy & Practice*, *5*(1), 7-74.

Boaler, J. (2016). Designing mathematics classes to promote equity and engagement. *The Journal of Mathematical Behavior*, *100*(41), 172-178.

Boaler, J. (2000). Mathematics from another world: Traditional communities and the alienation of learners. *The Journal of Mathematical Behavior*, *18*(4), 379-397.

Boaler, J., & Greeno, J. G. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple Perspectives on Mathematics Teaching and Learning*, Elsevier Science.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, *110*(3), 608-645.

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher*, *33*(8), 3-15.

Borko, H., Carlson, J., Deutscher, R., Boles, K. L., Delaney, V., Fong, A., Jarry-Shore, M., Malamut, J., Million, S., & Mozenter, S. (2021). Learning to lead: An approach to mathematics teacher leader development. *International Journal of Science and Mathematics Education*, *19*(1), 121-143.

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education*, *24*, 417–436.

Brophy, J. E. (1999). *Teaching*. International Academy of Education and the International Bureau of Education.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, *18*(1), 32-42.

Burbules, N., & Rice, S. (1991). Dialogue across differences: Continuing the conversation. *Harvard Educational Review*, *61*(4), 393-417.

Charalambous, C. Y., & Delaney, S. (2020). Mathematics teaching practices and practice-based pedagogies: A critical review of the literature since 2000. In D. Potari & O. Chapman (Eds.), *The international handbook of mathematics teacher education: Volume 1 – Knowledge, beliefs, and identity in mathematics teaching and development* (pp. 355-390). Brill/Sense.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, *23*(7), 13-20.

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education*, *40*(1), 40-68.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3-4), 175-190.

Coles, A. (2013). Using video for professional development: the role of the discussion facilitator. *Journal of Mathematics Teacher Education*, *16*(3), 165-184.

Collins, A., Brown, J. S., & Newman, S. E. (1988). Cognitive apprenticeship: Teaching the craft of reading, writing and mathematics. *Thinking: The Journal of Philosophy for Children*, *8*(1), 2-10.

Cuoco, A., & McCallum, W. (2018). Curricular coherence in mathematics. In *Mathematics Matters in Education* (pp. 245-256). Springer.

Darling-Hammond, L., Flook, L., Cook-Harvey, C., Barron, B., & Osher, D. (2020). Implications for educational practice of the science of learning and development. *Applied Developmental Science*, *24*(2), 97-140.

Darling-Hammond, L., Hyler, M. E., & Gardner, M. (2017). *Effective teacher professional development.* Learning Policy Institute.

Dewey, J. D. (1929). *The quest for certainty: A study of the relation of knowledge and action.* Minton Blach & Co.

Desimone, L. M. (2011). A primer on effective professional development. *The Phi Delta Kappan, 92*(6), 68-71.

Desimone, L. M., & Garet, M. S. (2015). Best practices in teacher's professional development in the United States. *Psychology, Society, & Education*, *7*(3), 252-263.

DiNapoli, J., & Miller. E. K. (2022). Recognizing, supporting, and improving student perseverance in mathematical problem-solving: The role of conceptual thinking scaffolds. *Journal of Mathematical Behavior, 66,* 1-20. https://doi.org/10.1016/j.jmathb.2022.100965

DiNapoli, J., & Morales, Jr., H. (2021). Translanguaging to persevere is key for Latinx bilinguals' mathematical success. *Journal of Urban Mathematics Education, 14*(2), 71-104. https://doi.org/10.21423/jume-v14i2a390

Downton, A., & Sullivan, P. (2017). Posing complex problems requiring multiplicative thinking prompts students to use sophisticated strategies and build mathematical connections. *Educational Studies in Mathematics*, *95*(3), 303-328.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist*, *23*(2), 167-180.

Engle, R. A., & Conant, F. R. (2002). Guiding principles for fostering productive disciplinary engagement: Explaining an emergent argument in a community of learners classroom. *Cognition and Instruction*, *20*(4), 399-483.

Evans, R. (2012). Getting to no: Building true collegiality in schools. *Independent School, 71*(2), 99-107.

Flores, A. (2007). Examining disparities in mathematics education: Achievement gap or opportunity gap? *The High School Journal*, *91*(1), 29-42.

Forman, E. A. (1996). Learning mathematics as participation in classroom practice: Implications of sociocultural theory for educational reform. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 115-130). Lawrence Erlbaum Associates.

Fries, L., Son, J. Y., Givvin, K. B., & Stigler, J. W. (2021). Practicing connections: A framework to guide instructional design for developing understanding in complex domains. *Educational Psychology Review, 33*(2), 1-24.

Gallagher, H. A. (2016). *Professional development to support instructional improvement: Lessons from research*. SRI International. Retrieved from: https://www.sri.com/wp-content/uploads/2021/12/professional_development_to_support_instructional_improvement.pdf.

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, *38*(4), 915-945.

Goos, M., Galbraith, P., & Renshaw, P. (1999). Establishing a community of practice in a secondary mathematics classroom. In L. Burton (Ed.), *Learning mathematics: From hierarchies to networks* (pp. 36-61). The Falmer Press.

Granberg, C., Palm, T., & Palmberg, B. (2021). A case study of a formative assessment practice and the effects on students' self-regulated learning. *Studies in Educational Evaluation*, *68*, 100955. https://doi.org/10.1016/j.stueduc.2020.100955

Guskey, T. R. (2003). What makes professional development effective? *Phi Delta Kappan, 84*(10), 748-750.

Hand, V., & Gresalfi, M. (2015). The joint accomplishment of identity. *Educational Psychologist*, *50*(3), 190-203.

Herman, J., Epstein, S., Leon, S., Matrundola, D. L. T., Reber, S., & Choi, K. (2015). *Implementation and effects of LDC and MDC in Kentucky districts. Policy Brief No. 13.* National Center for Research on Evaluation, Standards, and Student Testing (CRESST).

Hiebert, J. (2013) *Conceptual and procedural knowledge: The case of mathematics*. Routledge.

Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher, 31*(5), 3-15.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second Handbook of Research on Mathematics Teaching and Learning*, *1*(1), 371-404.

Hiebert, J., & Stigler, J. W. (2017). Teaching versus teachers as a lever for change: Comparing a Japanese and a U.S. perspective on improving instruction. *Educational Researcher, 46*(4), 169-176.

Hodge, L. L., & Cobb, P. (2019). Two views of culture and their implications for mathematics teaching and learning. *Urban Education*, *54*(6), 860-884.

Horn, I. S., & Kane, B. D. (2015). Opportunities for professional learning in mathematics teacher workgroup conversations: Relationships to instructional expertise. *Journal of the Learning Sciences, 24*(3), 373-418.

Hung, M., Smith, W. A., Voss, M. W., Franklin, J. D., Gu, Y., & Bounsanga, J. (2020). Exploring student achievement gaps in school districts across the United States. *Education and Urban Society, 52*(2), 175-193.

Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction. *Journal for Research in Mathematics Education*, *44*(4), 646-682.

Kazemi, E., & Hubbard, A. (2008). New directions for the design and study of professional development: Attending to the coevolution of teachers' participation across contexts. *Journal of Teacher Education, 59*(5), 428-441.

Kelly, P. (2006). What is teacher learning? A socio-cultural perspective. *Oxford Review of Education*, *32*(4), 505-519.

Koellner, K., Jacobs, J., & Borko, H. (2011). Mathematics professional development: Critical features for developing leadership skills and building teachers' capacity. *Mathematics Teacher Education and Development*, *13*(1), 115-136.

LaMar, T., Leshin, M., & Boaler, J. (2020). The derailing impact of content standards–an equity focused district held back by narrow mathematics. *International Journal of Educational Research Open*, *1*, 100015. https://doi.org/10.1016/j.ijedro.2020.100015

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Leonard, H. S., DiNapoli, J., Murray, E., & Bonaccorso, V. D. (2022). Collegial frame processes supporting mathematics teacher learning in a community of practice. In the *American Education Research Association Online Paper Repository.* American Educational Research Association.

Lesseig, K., Elliott, R., Kazemi, E., Kelley-Petersen, M., Campbell, M., Mumme, J., & Carroll, C. (2017). Leader noticing of facilitation in videocases of mathematics professional development. *Journal of Mathematics Teacher Education*, *20*(6), 591-619.

Levya, L. A. (2021). Black women's counter-stories of resilience and within-group tensions in the white, patriarchal space of mathematics education. *Journal for Research in Mathematics Education, 52*(2), 117-151.

Martin, D. B. (2000). *Mathematics success and failure among African-American youth: The roles of sociohistorical context, community forces, school influence, and individual agency*. Lawrence Erlbaum Associates.

Martin, D. B. (2012). Learning mathematics while Black. *Educational Foundations, 26*(1/2), 47–66.

Mathematics Assessment Resource Service (MARS). (2015a). *Formative assessment lessons*. Mathematics Assessment Project. https://www.map.mathshell.org/lessons.php

Mathematics Assessment Resource Service (MARS). (2015b). *Representing linear and exponential growth*. Mathematics Assessment Project. https://www.map.mathshell.org/download.php?fileid=1732

Mathematics Assessment Resource Service (MARS). (2015c). *Representing quadratic functions graphically*. Mathematics Assessment Project. https://www.map.mathshell.org/download.php?fileid=1734

Milner IV, H. R. (2012). Beyond a test score: Explaining opportunity gaps in educational practice. *Journal of Black Studies*, *43*(6), 693-718.

Moses, R., & Cobb, C. E. (2001). *Radical equations: Civil rights from Mississippi to the Algebra Project*. Beacon Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Author.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Author.

National Council of Teachers of Mathematics. (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. Author.

National Council of Teachers of Mathematics. (2020). *Catalyzing change in middle school mathematics: Initiating critical conversations*. Author.

National Governors Association. (2010). *Common core state standards for mathematics*. National Governors Association Center for Best Practices and the Council of Chief State School Officers.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

Nelson, T. H., Deuel, A., Slavit, D., & Kennedy, A. (2010). Leading deep conversations in collaborative inquiry groups. *The Clearing House: A Journal of Educational Strategies, Issues and Ideas*, *83*(5), 175-179.

Porter, A., McMaken, J., Hwang, J., & Yang, R. (2011). Common core standards: The new U.S. intended curriculum. *Educational Researcher, 40*(3), 103-116.

Rittle-Johnson, B., Siegler, R.S., & Alibali, M.W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346–362.

Rosli, R., & Aliwee, M. F. (2021). Professional development of mathematics teacher: A systematic literature review. *Contemporary Educational Research Journal*, *11*(2), 81-92.

Ross, J., & Bruce, C. (2007). Professional development effects on teacher efficacy: Results of randomized field trial. *The Journal of Educational Research, 101*(1), 50-60.

Santagata, R., & Bray, W. (2016). Professional development processes that promote teacher change: The case of a video-based program focused on leveraging students' mathematical errors. *Professional Development in Education, 42*(4), 547-568.

Schildkamp, K., van der Kleij, F. M., Heitink, M. C., Kippers, W. B., & Veldkamp, B. P. (2020). Formative assessment: A systematic review of critical teacher prerequisites for classroom practice. *International Journal of Educational Research*, *103*, 101602. https://doi.org/10.1016/j.ijer.2020.101602

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of U.S. mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*(5), 525-559.

Schoenfeld, A. H. (2023). A theory of teaching. In A.K. Praetorius & C. Y. Charalambous (Eds.), *Theorizing teaching: Current status and open issues* (pp. 159–187). Springer.

Schoenfeld, A. H. (2013). Classroom observations in theory and practice. *ZDM Mathematics Education*, *45*, 607-621. https://doi.org/10.1007/s11858-012-0483-1

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. *Educational Researcher*, *43*(8), 404-412.

Schoenfeld, A. H. (2015). Thoughts on scale. *ZDM Mathematics Education*, *47*(1), 161-169. https://doi.org/10.1007/s11858-014-0662-3

Schoenfeld, A. H. (2017). Use of video in understanding and improving mathematical thinking and teaching. *Journal of Mathematics Teacher Education*, *20*(5), 415-432.

Schoenfeld, A.H., Fink, H., Sayavedra, A., Weltman, A., & Zuñiga-Ruiz, S. (2023). *Mathematics teaching on target: A guide to teaching for robust understanding at all grade levels*. Routledge.

Schoenfeld, A. H., Floden, R. E., & the Algebra Teaching Study and Mathematics Assessment Project. (2014). *The TRU Math Scoring Rubric*. Graduate School of Education, University of California, Berkeley & College of Education, Michigan State University. Retrieved from http://ats.berkeley.edu/tools.html.

Schoenfeld, A. H., & the Teaching for Robust Understanding Project. (2016). *An introduction to the Teaching for Robust Understanding (TRU) framework*. Graduate School of Education. http://map.mathshell.org/trumath.php

Selkrig, M., & Keamy, K. (2015). Promoting a willingness to wonder: Moving from congenial to collegial conversations that encourage deep and critical reflection for teacher educators. *Teachers and Teaching*, *21*(4), 421-436.

Shepard, L. A. (2000). *The role of classroom assessment in teaching and learning* (Report No. CSE-TR-517). Office of Educational Research and Improvement. https://cresst.org/wp-content/uploads/TECH517.pdf

Smith, R., Ralston, N. C., Naegele, Z., & Waggoner, J. (2020). Team teaching and learning: A model of effective professional development for teachers. *Professional Educator, 43*(1), 80-90.

Smith, M. S., & Stein, M. K. (2018). *5 practices for orchestrating productive mathematics discussions*. National Council of Teachers of Mathematics.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455-488.

Stigler, J. W., & Hiebert, J. (1997). Understanding and improving classroom mathematics instruction: An overview of the TIMSS video study. *Phi Delta Kappan*, *79*(1), 14.

Stigler, J. W., & Hiebert, J. (2004). Improving mathematics teaching. *Educational Leadership*, *61*(5), 12-17.

Sztajn, P., Borko, H., & Smith, T. (2017). Research on mathematics professional development. In J. Cai's (Ed.) *Compendium for Research in Mathematics Educatio*n (pp. 793-823). National Council of Teachers of Mathematics.

Tekkumru-Kisa, M., Stein, M. K., & Doyle, W. (2020). Theory and research on tasks revisited: Task as a context for students' thinking in the era of ambitious reforms in mathematics and science. *Educational Researcher, 49*(8), 606-617.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education, 24*(2), 244-276.

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education, 18*(4), 375-400.

Wenger, E. (1998a). Communities of practice: Learning as a social system. *Systems Thinker*, *9*(5), 2-3.

Wenger, E. (1998b). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press.

Yoon, K. S., Duncan, T., Lee, S. W. Y., Scarloss, B., & Shapley, K. L. (2007). *Reviewing the evidence on how teacher professional development affects student achievement* (Issues & Answers Report, REL 2007–No. 033). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Southwest. http://ies.ed.gov/ncee/edlabs

Zepeda, S. J. (2020). Standards of collegiality and collaboration. In S. Gordon (Ed.), *Standards for instructional supervision: Enhancing teaching and learning* (pp. 63-75). Routledge.

APPENDIX A

Detailed Example of the Coding Involved in Our Analysis Plan

In this PLC session, teacher participants were discussing a video in which students were struggling to (i) interpret speed as the slope of a linear graph and (ii) translate between the equation of a line and its graphical representation. In each transcript example below, we color coded the teacher's **diagnosis (red)**, **prognosis (green)**, and **motivation (blue)**. We also highlighted frame processes according to the color scheme in the Frame Alignment Processes in Table 1. Teacher A began the discussion below; Table 3 summarizes how we coded Teacher A's frame.

Teacher A: So I know one thing that I'm doing now, when we have word problems with, like, a situation like this . . . I'm seeing that **a lot of students don't really understand the word problem**...I've learned by doing that in math when they have a word problem before they even start thinking about the "math" that's in the word problem, it helps them to understand what's going on, like if I just asked, "Who are the characters? What's the conflict?" or "What's the problem? What's the goal at the end? What are they trying to figure out?" And I think that, **it's actually like a list of, like a break sheet of questions that they have to fill out before they actually start solving the problem.**

Table 3

Summary of Coding Teacher A's Frame

Category	Code & Explanation
Diagnosis	Some students do not understand the word problem in the lesson.
Prognosis	The teacher could give students comprehension questions about the situation in the word problem.
Motivation	None
Frame Process	This frame was coded as an articulating frame because this is the first time that this problem of practice is addressed in this PLC meeting.
TRU Dimension	This frame was coded as aligned to **Cognitive Demand** because the suggested teacher move involved scaffolding the task in a way to help create and maintain an environment of productive intellectual challenge.
TRU Rubric Score	This frame was scored as **2.5** because although asking comprehension questions could help students engage with the word problem and does not remove opportunities for productive struggle, it is unclear how such opportunities could help students build understanding of central mathematical ideas or engage in mathematical practices.

Teacher B then built on Teacher A's ideas in the following **connected** frame. See Table 4 for a summary of how we coded Teacher B's frame.

Teacher B: ...It's like you **do a first read** and you just identify what is the story about. You **do a second read** and you identify what are the quantities and their relationships. Like, what are the numbers and what do they mean in the situation. And then **the third read** is you try to ponder, what question are they going to ask me without knowing the question. **So then that way you're being a problem solver before the problem is already presented to you.**

Table 4

Summary of Coding Teacher B's Frame

Category	Code & Explanation
Diagnosis	Some students do not understand the word problem in the lesson (the same diagnosis as the connected frame).
Prognosis	Teachers could encourage three reads of the word problem: to identify what the story is about, to identify what the numbers mean in context, and to predict/pose the question to be asked.
Motivation	Requiring students to predict what question the problem is asking before reading it will engage students in the process of problem solving before the problem is officially presented to them, which will help them engage more deeply.

Frame Process	This frame was coded as a transforming frame because Teacher B's suggested move leveraged Teacher A's suggestion to generate a new understanding about scaffolding. The motivation provided by Teacher B makes it clear that the "three reads" will not only help students engage with the word problem, but will provide them with an important opportunity to develop their problem-solving practices.
TRU Dimension	This frame was coded as aligned to the **Cognitive Demand** dimension because the suggested teacher move built on the previous frame, involving scaffolding the task in a way to help create and maintain an environment of productive intellectual challenge.
TRU Rubric Score	This frame was scored as a **3** because the "third read" will support students in productively struggling to make connections between the word problem and the mathematical ideas central to the problem situation.

APPENDIX B

Rubric for TRU Talk in PLCs

Rubric for TRU Talk in PLCs

The purpose of this rubric is to capture the alignment of PLC teachers' talk with the TRU Framework dimensions and the rigor of such PLC teachers' talk.

The Mathematics	Cognitive Demand	Equitable Access to Mathematical Content	Agency, Ownership, and Identity	Formative Assessment
To what extent does PLT teacher talk focus on their own understandings of the accuracy, coherence, and justification of the mathematical content?	*To what extent does PLT teacher talk focus on classroom interactions that create and maintain an environment of productive intellectual challenge that is conducive to students' mathematical development?*	*To what extent does PLT teacher talk focus on supporting all students in equal access to and meaningful participation with the mathematics?*	*To what extent does PLT teacher talk focus on providing students opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to students' development of agency, ownership, and their identities as doers of*	*To what extent does PLT teacher talk focus on monitoring and helping students to refine their thinking?*
1: The mathematics discussed is not at grade level, or discussions are aimed at "answer getting." Explanations, if they appear, are largely procedural.	1: PLT teachers suggest and/or agree with classroom activity or teacher intervention that constrains students to activities such as applying straightforward or memorized procedures.	1: PLT teachers suggest and/or agree with classroom activity that leaves some students disengaged or marginalized, and differential access to the mathematics or to the group is not being addressed.	1: PLT teachers suggest and/or agree with classroom activity or teacher interventions that either constrain students to producing short responses to the teacher OR do not address clear imbalances in group discussions.	1: PLT teachers suggest and/or agree with classroom activities that are simply corrective (e.g., leading students down a predetermined path) and the teacher does not meaningfully solicit or pursue student thinking.
2: Discussions are at grade level but are primarily skills oriented, with few opportunities for making connections (e.g., between procedures and concepts) or for mathematical coherence.	2: PLT teachers suggest moves (e.g., hints or scaffolds) and/or agree with classroom activity that offers possibilities of productive engagement or struggle with central mathematical ideas but scaffolds away some challenges and/or removes some opportunities for productive struggle with central mathematical ideas and/or engagement in mathematical practices.	2: PLT teachers suggest moves and/or agree with classroom activity that illustrates some efforts to provide mathematical access to a wide range of students; OR the teacher does not support student participation in group activities like student-to-student discussion.	2: PLT teachers suggest moves and/or agree with classroom activity that allow at least one student to talk about the mathematical content, but the teacher is still the primary driver of conversations and arbiter of correctness OR students are not supported in building on each other's ideas.	2: PLT teachers suggest moves and/or agree with classroom activity that solicit student thinking, but subsequent classroom discussion does not build on nascent ideas. The moves and/or classroom activity are corrective in nature, possibly by leading students in the "right" directions.
3: Explanations of and justifications for central grade level mathematical ideas are coherent.	3: PLT teachers suggest specific moves (e.g., hints or scaffolds) and/or agree with classroom activity that support students in productive struggle in building understandings of central mathematical ideas and engaging in mathematical practices without scaffolding away challenges.	3: PLT teachers suggest specific moves and/or agree with classroom activity that would actively support and to some degree achieve broad and meaningful participation from all students; OR to establish participation structures that result in such engagement.	3: PLT teachers suggest specific moves and/or agree with classroom activity that allow at least one student to put forth and defend their ideas/reasoning; AND students build on each other's ideas OR the teacher ascribes ownership for students' ideas in subsequent discussion.	3: PLT teachers suggest specific moves and/or agree with classroom activity that solicits student thinking, AND plans for subsequent classroom discussion to respond to those ideas, by building on the productive beginnings or addressing possible misunderstandings.

We adapted, with permission, Schoenfeld et al.'s (2014) TRU Math Rubric to fit our context of PLC teachers' talk. See Table 5 for an example of different ratings for a sample TRU dimension.

Table 5
Example of TRU Talk Ratings for Formative Assessment

TRU Talk Rating	Formative Assessment Example
1	"The teacher could correct the student's matching mistake and show them how to correctly match cards in the card sort."
1.5	"The teacher could ask a student to share their thinking about a match they made and then show them how to correctly match cards in the card sort."
2	"The teacher could elicit student thinking by giving them blank cards and asking them to create their own word problem scenario similar to those in the activity."
2.5	"The teacher could elicit student thinking by giving them blank cards and asking them to create their own word problem scenario similar to those in the activity. Then, the teacher could work out one of the problems on the board."
3	"The teacher could elicit student thinking by giving them blank cards and asking them to create their own word problem scenario similar to those in the activity. Then, the teacher could facilitate a whole-class discussion about these scenarios to build on students' thinking and address any misunderstandings."

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